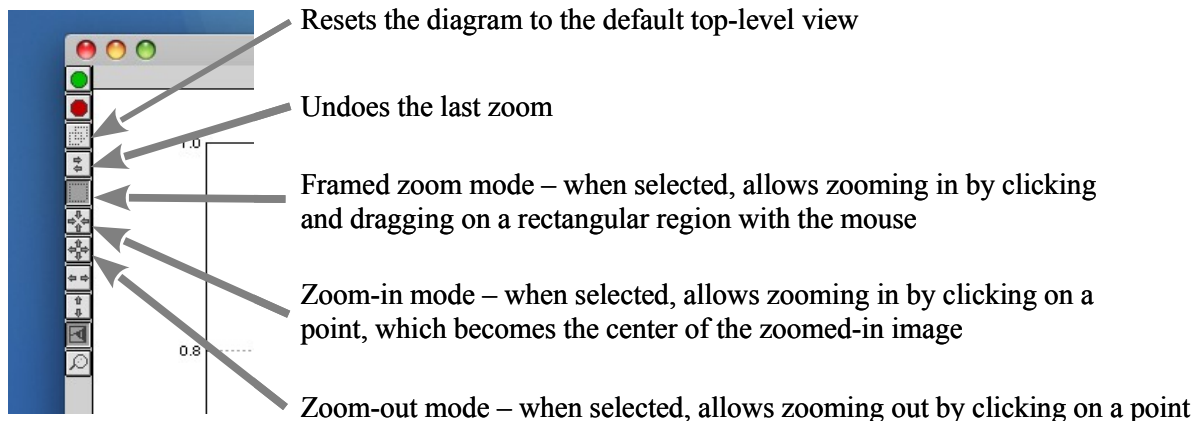## Lab 4: The Bifurcation Diagram

In this lab you will explore the bifurcation diagram for the logistic map. Write down your answers to each of the starred (**) exercises as you go. You will submit these answers with your next homework assignment.

1. Click on the *Chaos* icon in the Dock, which will open the *Exploring Chaos* program, then click on the *Bifurcation Diagrams* button. This will open a window with a bifurcation diagram for the logistic map $x_{t+1} = R\,x_t(1 - x_t)$. Maximize the window so that it fills the entire screen. The R parameter ranges from 2.8 to 4.0 on the horizontal axis. For each R value, the vertical axis shows the corresponding attractor points – that is, the final set of x values that the equation converges to when iterated for a large number of steps, starting from any initial x value in the range 0 to 1.

   As you mouse over the diagram, the cursor coordinates appear at the top of the window. For example, the long-term behavior of the equation with R set to 3.2, starting from any initial x value, eventually settles into a period-two limit cycle, continually bouncing back and forth between 0.8 and 0.51. Move your mouse to these points on the diagram and verify their coordinates.

2. ** Now close the bifurcation diagram and click on the *Iterations (1d)* applet button. This will open a window that displays a single trajectory (also called an "orbit") of the logistic map iterated over many time steps, starting from a particular value of x. Maximize the window so that it fills the entire screen. Change R to 3.2 by typing the new value into the *Parameter* field in the top right corner of the window (you must also click on the green button in the top left corner in order for the new value to take effect). The graph will show what happens when the equation is iterated for 200 time steps with $R = 3.2$, starting from $x = 0.3$. Notice that, like before, the cursor coordinates appear at the top of the window as you move the mouse around. You can use this to find the values of x at different time steps in the trajectory. How many time steps does it take for the trajectory to settle into periodic behavior? What are the final attractor values?

3. ** By changing the *Parameter* and *Initial value* fields, you can view the long-term behavior of the logistic map for any R setting and initial value of x. Change the *Initial value* field to a few other values in the range 0 to 1. Does the equation always converge to the same two attractor values when $R = 3.2$? Try starting from 0.9, 0.5, 0.3, 0.2, 0.1, 0.001, and 0.0000001. What happens when you start from exactly 0 or exactly 1? Explain.

4. Close the trajectory window and reopen the bifurcation diagram. You can zoom in on any portion of the diagram by simply clicking and dragging the mouse, as long as the "Framed zoom mode" button is selected on the left toolbar (see below). Several other useful buttons are also shown. As you mouse over each button, a brief explanation of its function appears at the top of the window. Try zooming in and out of several different regions until you feel comfortable using the navigation controls to explore the diagram.

Resets the diagram to the default top-level view

Undoes the last zoom

Framed zoom mode – when selected, allows zooming in by clicking and dragging on a rectangular region with the mouse

Zoom-in mode – when selected, allows zooming in by clicking on a point, which becomes the center of the zoomed-in image

Zoom-out mode – when selected, allows zooming out by clicking on a point

5. ** Whereas the *Iterations (1d)* applet shows a single trajectory for a particular value of R, the bifurcation diagram summarizes the long-term behavior of the logistic map equation for many different values of R. The *Initial value* parameter in the top right corner of the bifurcation diagram window specifies the initial value of x that is used to compute the equation's long-term behavior for every value of R shown. The actual starting value is not that important, since the equation converges to the same set of attractor values in the long run, although for some starting values the equation may take longer to settle into its final behavior (fixed, periodic, or chaotic) than for others. Try out different initial value settings between 0 and 1 to see if the resulting set of attractor values changes, or takes longer to compute. What happens when exactly 0 or exactly 1 is used? Explain.

6. To accurately compute the set of attractor points for a particular R value, the program needs to wait long enough for the equation to settle into its long-term pattern of behavior (fixed, periodic, or chaotic). The *Discard first* parameter setting in the top right corner specifies how many trajectory values to ignore before plotting the equation's behavior. In general, a higher setting will give more accurate results, but will take longer to compute. Reset the diagram to the top level and try changing *Discard first* to 10, 100, 1000, and 10000 to see the effect. Notice that the bifurcation points appear "blurry" at lower settings. This is even more apparent at higher zoom magnifications. For best results, you should use a high setting (preferably 10000 or greater) whenever possible.

7. The *Show next* setting controls how many vertical points are plotted for each value of R once the corresponding equation has settled into its long-term behavior (after ignoring the initial number of trajectory values specified by *Discard first*). These points represent the set of attractor values that the trajectory has settled into. If the long-term behavior is fixed or periodic, this will be a finite set of values, but if the behavior is chaotic, the number of values is in principle infinite (though only a finite number of them can be plotted in the diagram), and the attractor is called a "strange attractor". In general, choosing a higher setting for *Show next* will give a more accurate picture of the bifurcation diagram. Reset the diagram to the top level and try settings of 10, 100, 1000, and 10000. Higher settings can be used to increase the contrast between regions of order and chaos in the diagram, such as the "islands" of stable periodic behavior embedded within the chaotic region above $R > 3.6$.

8. ** The first bifurcation occurs at an R value of 3.0. When the "knob" reaches this value, the long-term behavior of the equation changes from fixed to periodic. That is, the single fixed point "splits" into two periodic points. What is the value of the fixed point at this bifurcation?

9. ** As R is increased further, the two periodic attractor points slowly drift apart. When $R = 3.4$, what are the values of the two attractor points? How many steps does it take for an initial value of $x = 0.3$ to fully converge to the attractor? What about an initial value of 0.2? Try a few other values as well.

10. ** These periodic attractor points are *stable* in the sense that any nearby x values will quickly converge to the attractor, as you saw in the previous question. However, the original fixed point, from which the two periodic points were "born", still exists. However, it has now become *unstable*, meaning that although its precise value stays perfectly fixed when iterated, any slight deviation from this value, no matter how tiny, will cause the subsequent values to quickly run away from the fixed point (indeed, they will head for the periodic attractor!). You can view unstable fixed points by clicking on the bottom button in the left toolbar of the bifurcation diagram.

    What is the approximate value of the unstable fixed point when $R = 3.4$? Starting from this value, about how many steps does it take to reach the periodic attractor? Can you make the starting value more precise (i.e. closer to the true unstable fixed point value) in order to make the values "hang around" the fixed point a little longer?

11. ** In fact, there is no way to prevent the inevitable divergence, because we simply can't specify the starting value to infinite precision. (This is a little like trying to balance a pencil on its sharpened end.) Even starting with $x = 0.7058823529411764$, which is quite close to the true fixed point value, we still eventually diverge from it. About how many steps does it take to reach the periodic attractor in this case?

12. ** Using the logistic map bifurcation diagram, find values of R that give rise to the following long-term behavior:

- a fixed point

- a period-2 limit cycle

- a period-3 limit cycle

- a period-4 limit cycle

- a period-5 limit cycle

- a period-6 limit cycle

- a period-7 limit cycle

- a period-8 limit cycle

- a period-9 limit cycle

- chaotic behavior

13. ** For the **period-3** attractor you found in the question 12, about how many iterations are necessary in order for a trajectory to reach the attractor, starting from the indicated values of x?

- starting from $x = 0.1$

- starting from $x = 0.2$

- starting from $x = 0.3$

- starting from $x = 0.4$

- starting from $x = 0.5$

- What happens if you start from $x = 0$ exactly?

14. ** For the **period-6** attractor you found in the question 12, about how many iterations are necessary in order for a trajectory to reach the attractor, starting from the indicated values of x?

- starting from $x = 0.1$

- starting from $x = 0.5$

- starting from $x = 0.7$

15. ** Using the logistic map bifurcation diagram, record the R values at which the first few bifurcations occur as R is increased. Use these values to calculate an estimate of Feigenbaum's constant. Then zoom into the diagram to locate another region of period-doubling behavior. Record a few R values from this region and repeat your calculation. How well do the values agree? (Hint: setting the *Discard first* parameter to a high value such as 100000 will increase the accuracy of the R values.)