A new kind of number

Pythagoras believed that *all quantities* could be expressed as the ratio of two whole numbers in lowest terms. (A whole number itself can be written as a ratio with denominator 1, such as $\frac{5}{1}$.)

Now, consider a square of unit area, with sides equal to 1. What is the length of its diagonal?

By the Pythagorean theorem, $diagonal^2 = 1^2 + 1^2$, so $diagonal = \sqrt{2}$.

Pythagoras assumed that the square root of 2, like any quantity, must be exactly expressible as the ratio of two whole numbers a and b in lowest terms. Let's see where this assumption leads us:

$\frac{a}{b} = \sqrt{2}$, with $\frac{a}{b}$ in lowest terms

so $\frac{a^2}{b^2} = 2$, by squaring both sides

so $a^2 = 2b^2$, by moving the b^2 to the right-hand side

so a^2 must be *even*, because 2 times anything gives an even number

so a itself must be *even*, because we know that squaring preserves even/odd-ness

that is, $a = 2x$ for some other number x

so $a^2 = 4x^2$, by squaring both sides

so $2b^2 = 4x^2$, since we already established that $a^2 = 2b^2$

so $b^2 = 2x^2$, by dividing both sides by 2

so b^2 must be *even*, because 2 times anything gives an even number

so b itself must be *even*, because we know that squaring preserves even/odd-ness

so a and b must *both* be even — but this contradicts our starting assumption!

so our starting assumption *must be wrong*, because each step of the reasoning is indisputably correct, but it leads straight to a logical contradiction

that is, $\sqrt{2}$ *cannot* be expressed as a ratio of whole numbers!

So there *must exist* other kinds of numbers — *irrational* numbers — that express quantities such as the length of the diagonal of a simple unit square! This discovery deeply disturbed Pythagoras and his followers, because it undermined their entire worldview, which held that the mathematics of whole numbers was perfect and complete. But the logical conclusion cannot be avoided, and we are forced to expand our concept of *number* as a result.