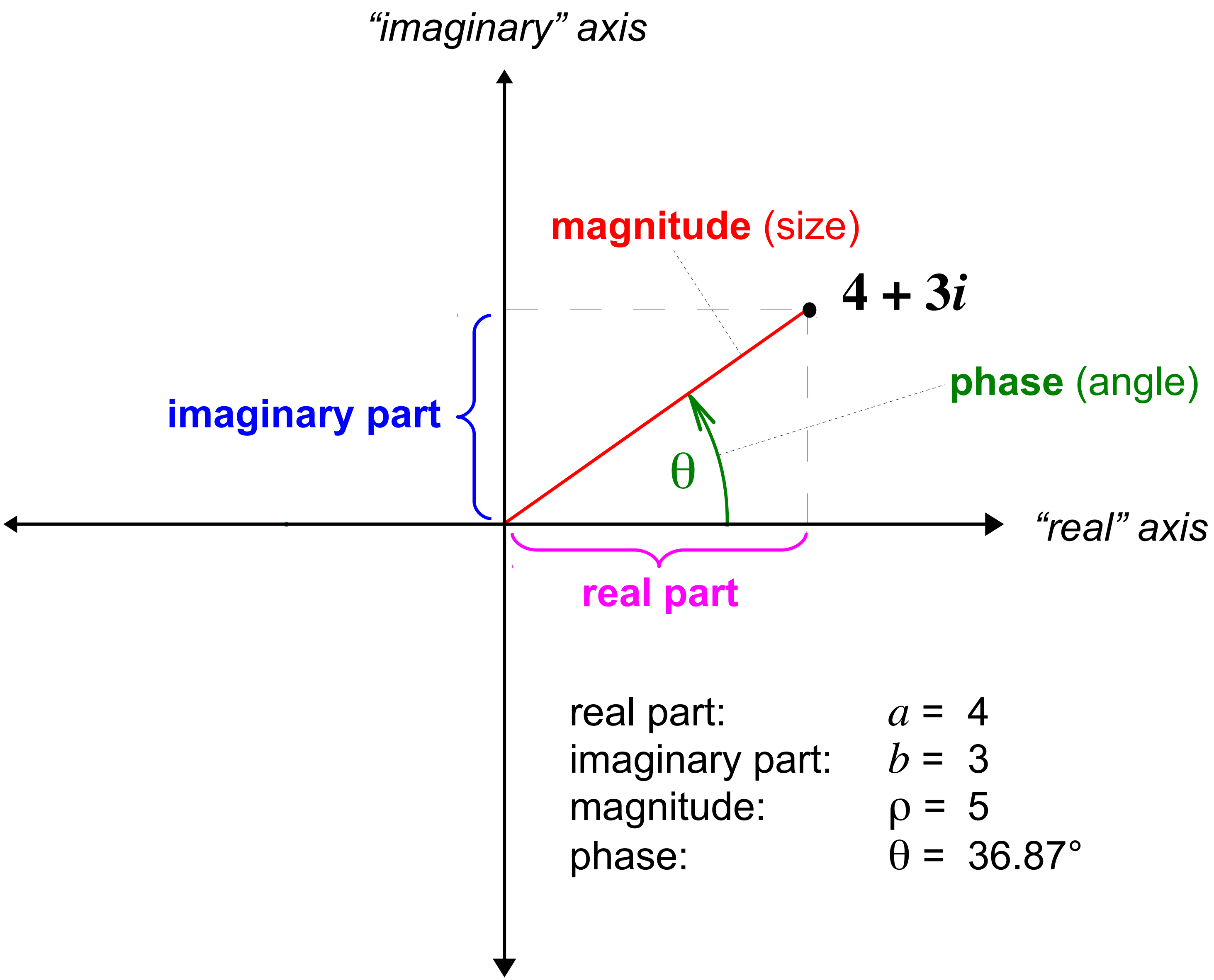
Complex Numbers



real part:	$a =$ 4
imaginary part:	$b =$ 3
magnitude:	$\rho =$ 5
phase:	$\theta =$ 36.87°

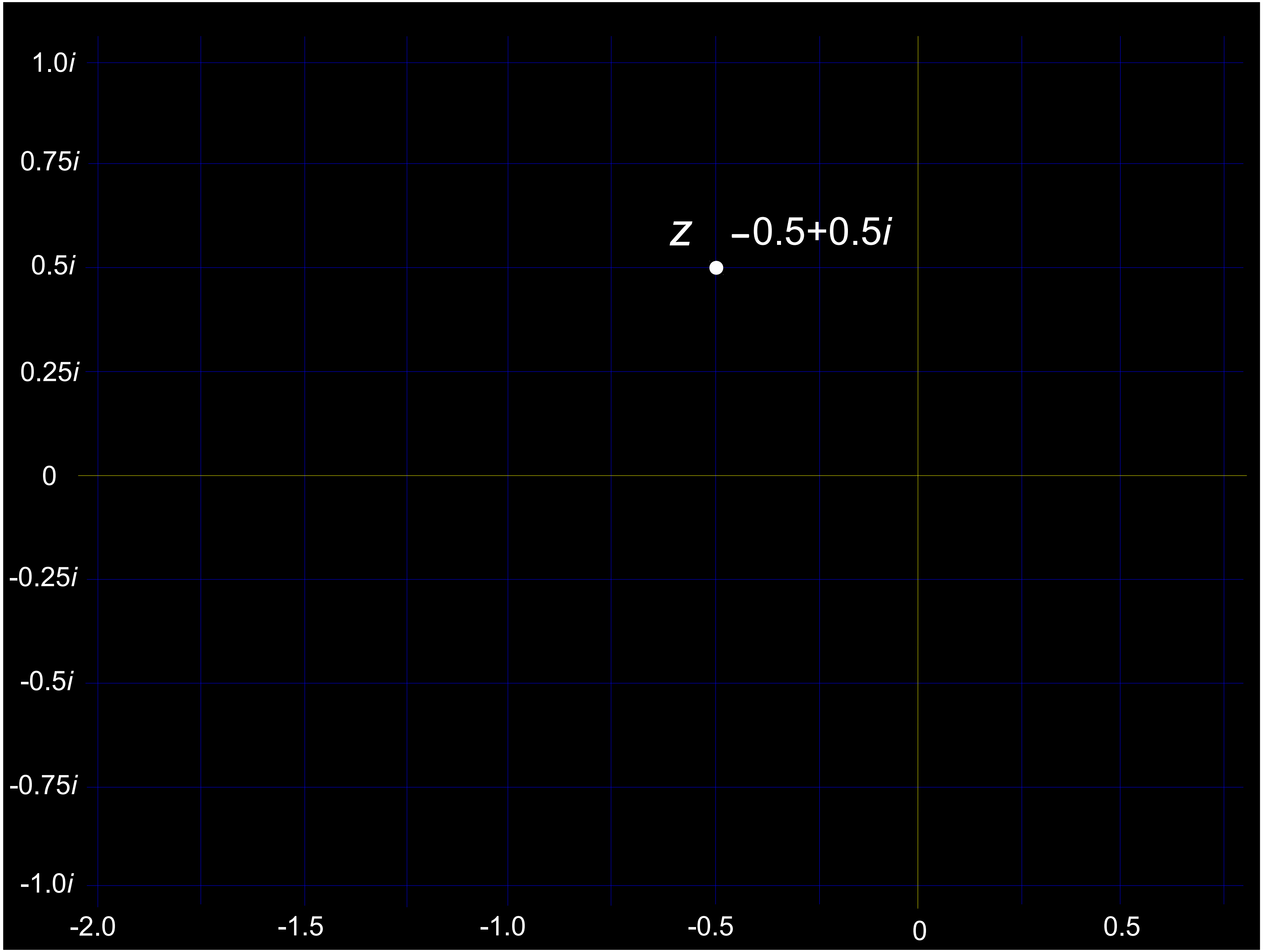
Complex Arithmetic

- To **add** two complex numbers, you just:
  - Add their real parts
  - Add their imaginary parts

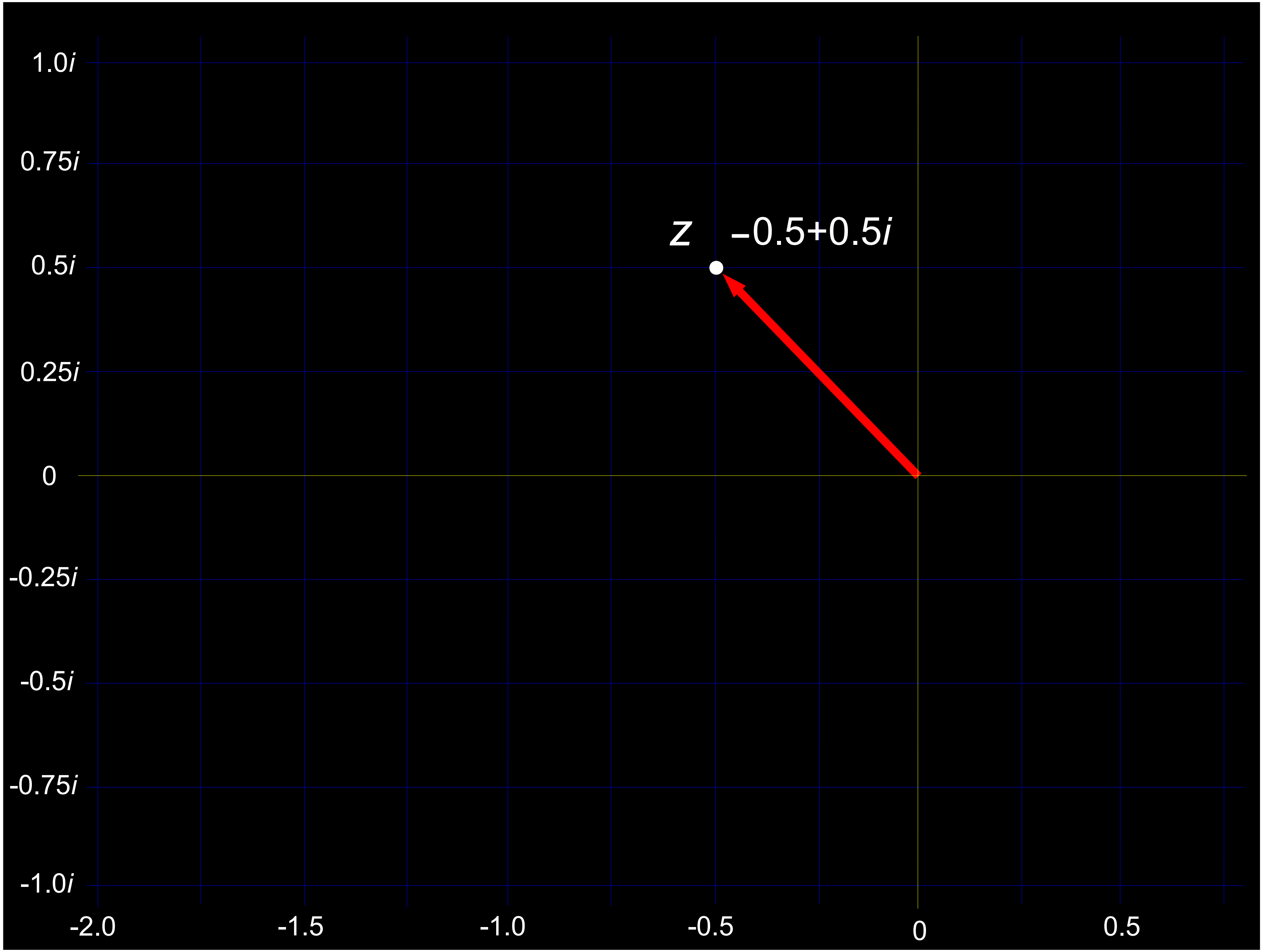
- To **multiply** two complex numbers, you just:
  - Multiply their magnitudes
  - Add their phases (angles)

- To **square** a complex number, you just:
  - Square its magnitude
  - Double its phase (angle)

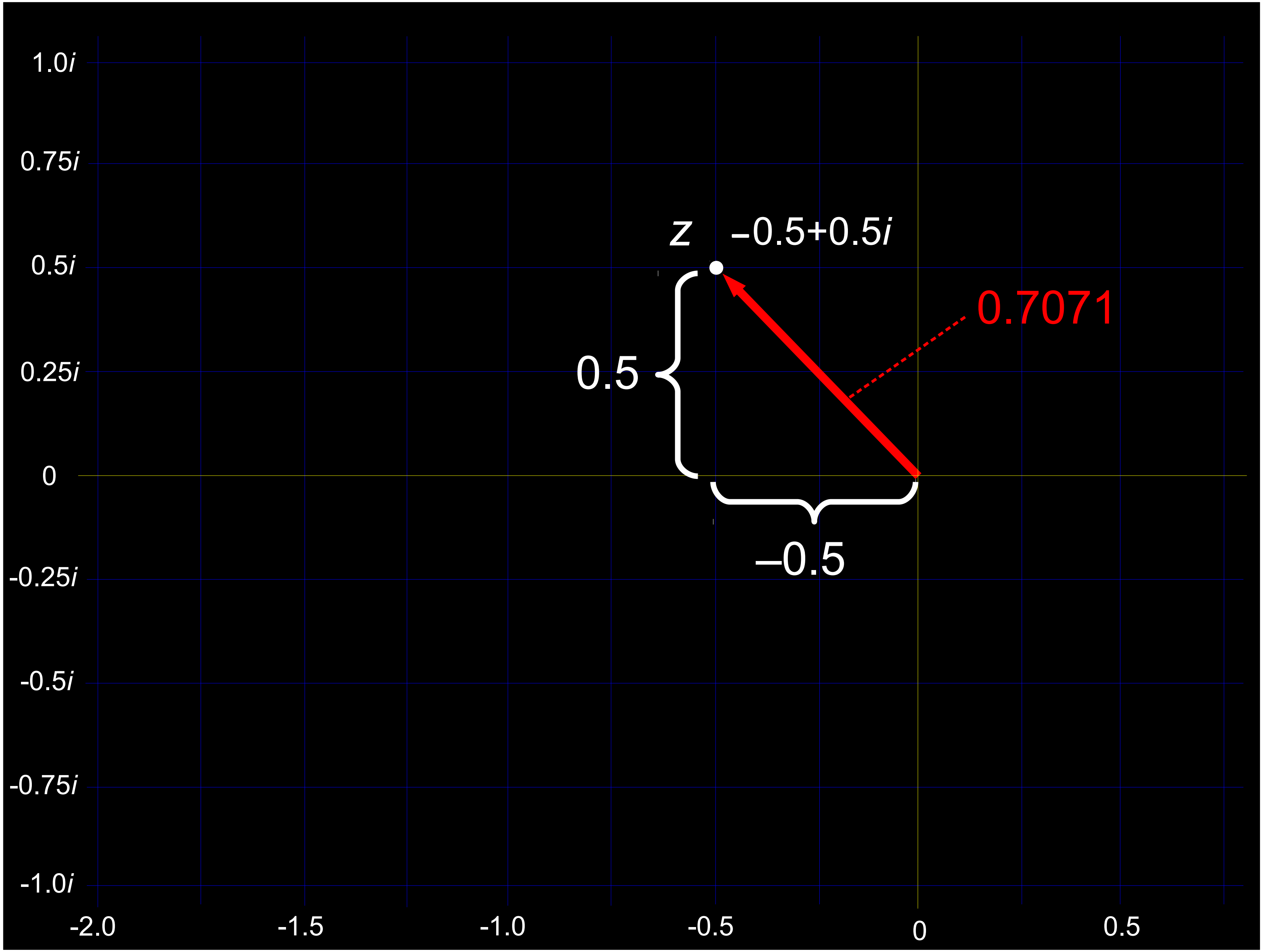
The Complex Plane



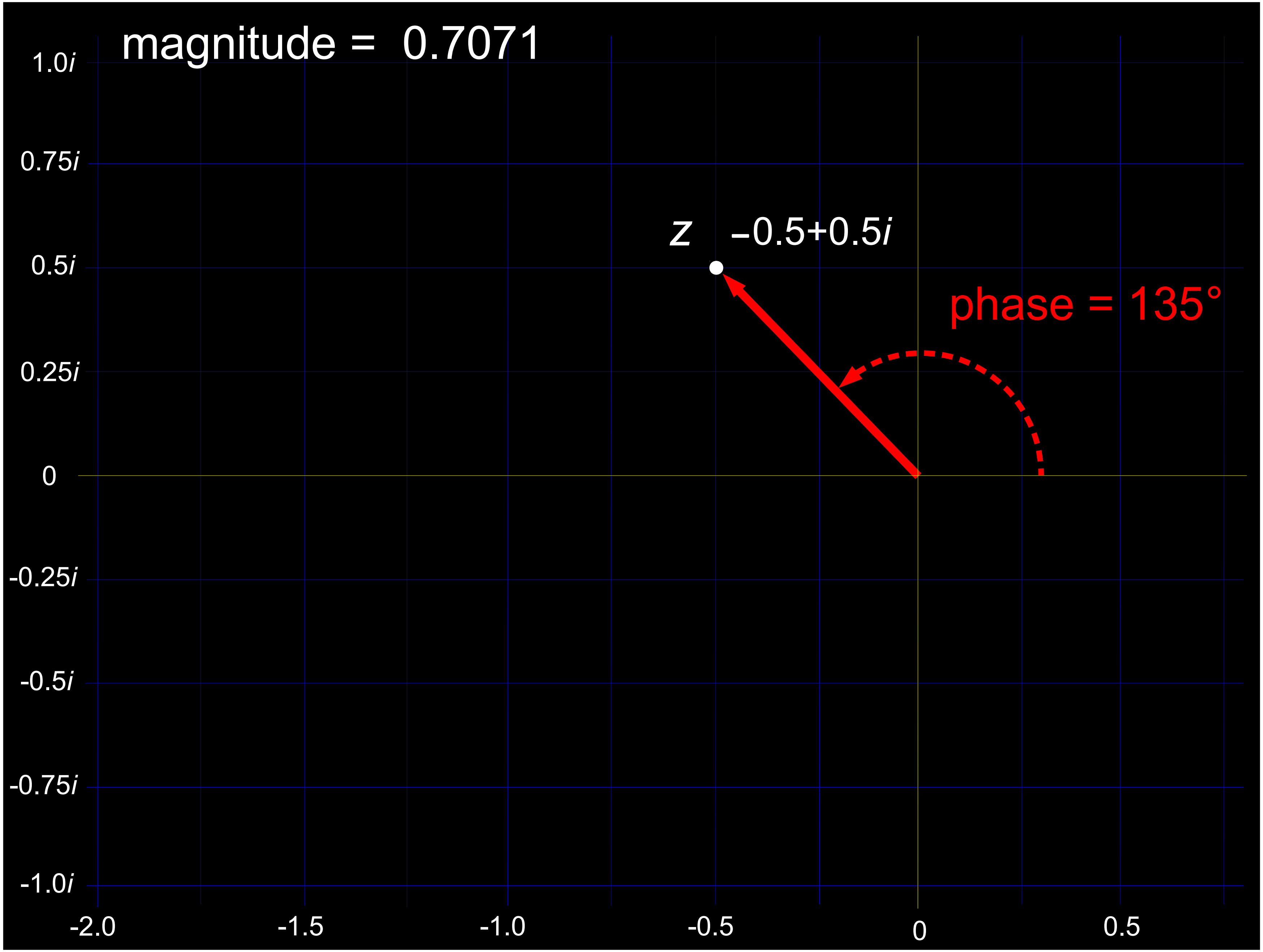
What is its Magnitude?



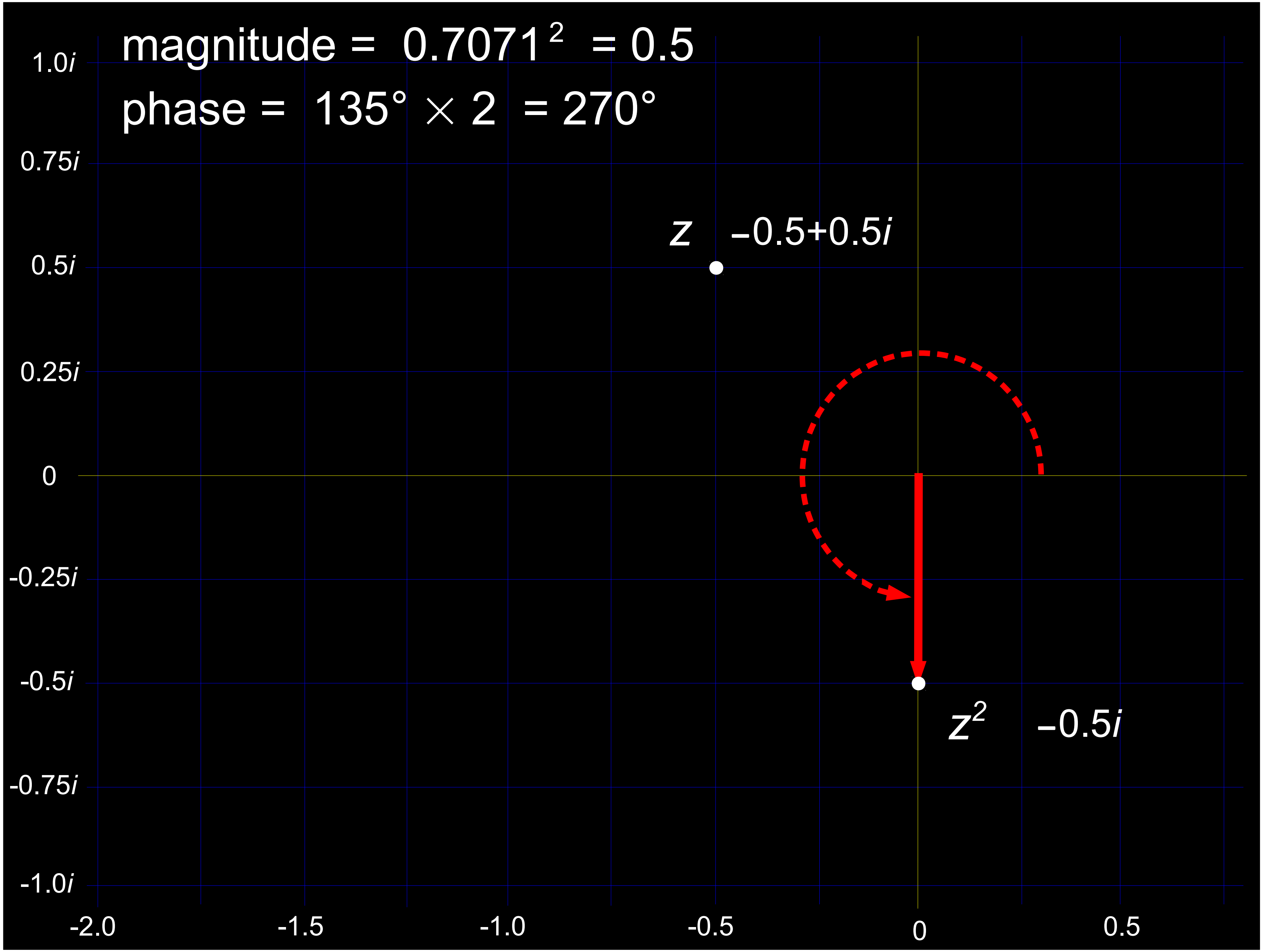
Square root of $(-0.5)^2 + (0.5)^2 = 0.7071$



What is its Phase?

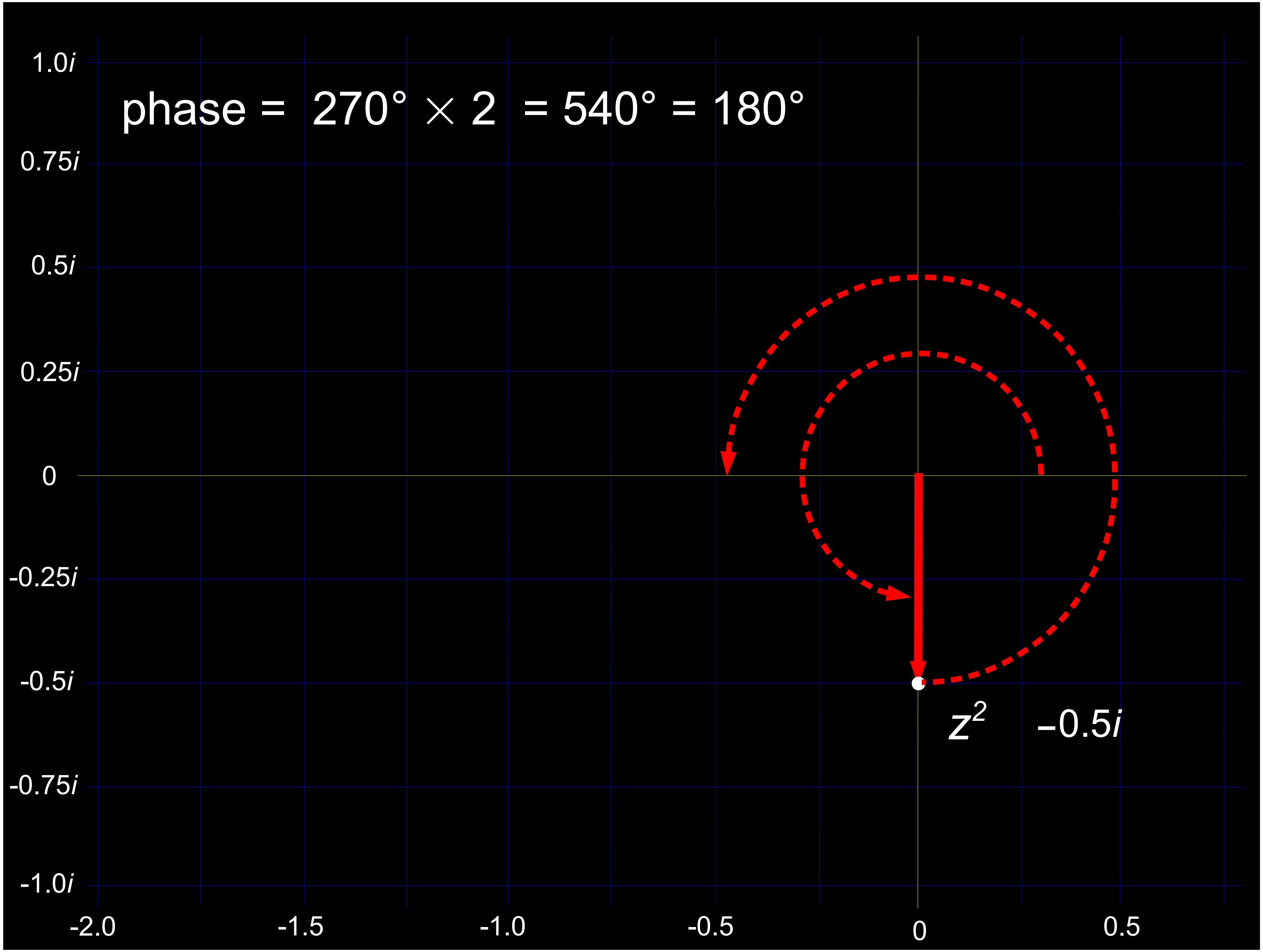


What is z^2 ?

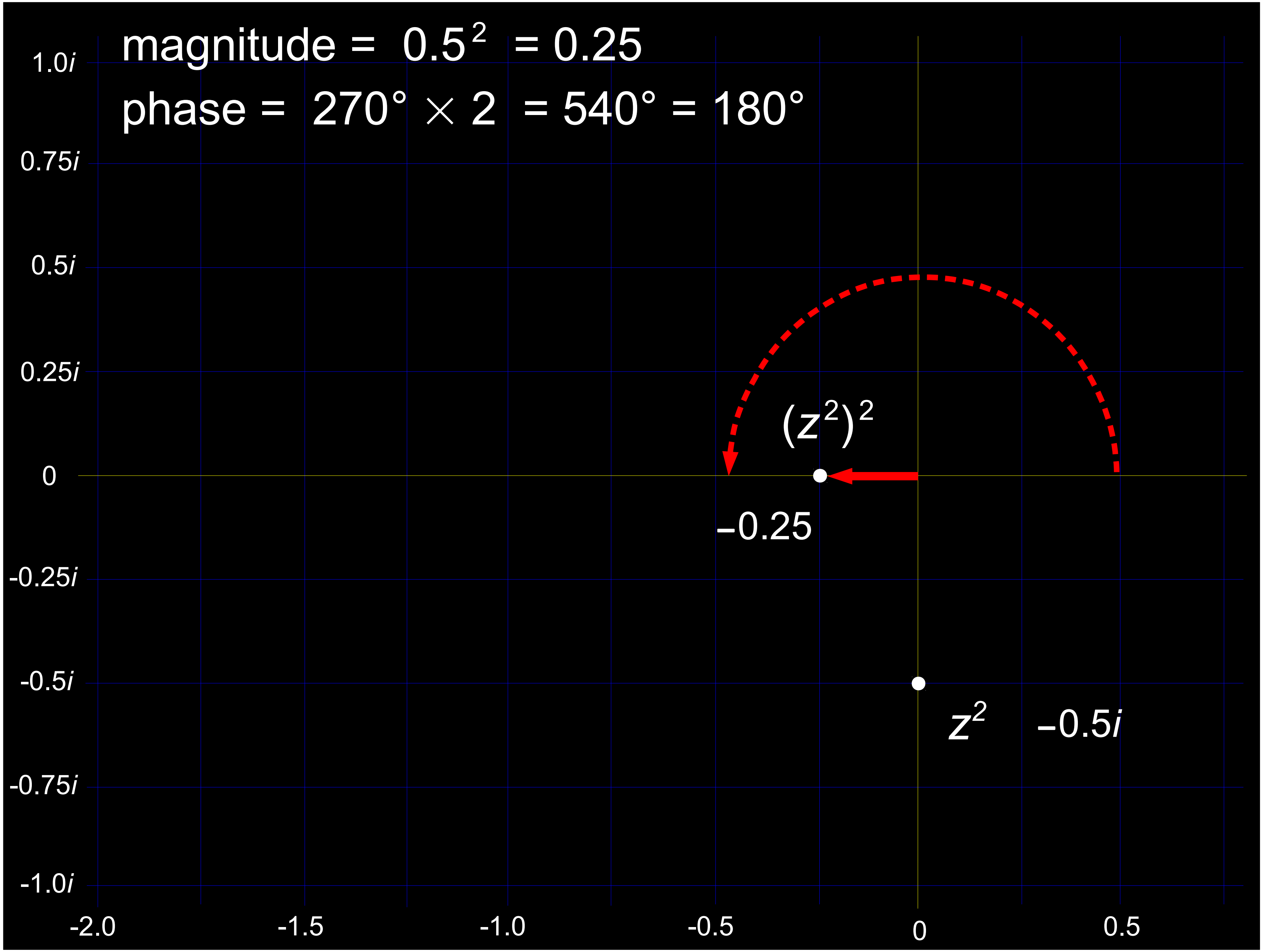


magnitude = 0.7071 ^2 = 0.5
phase = 135° × 2 = 270°
1.0i
0.75i
0.5i
0.25i
0
-0.25i
-0.5i
-0.75i
-1.0i
-2.0
-1.5
-1.0
-0.5
0
0.5
z −0.5+0.5i
z^2 −0.5i

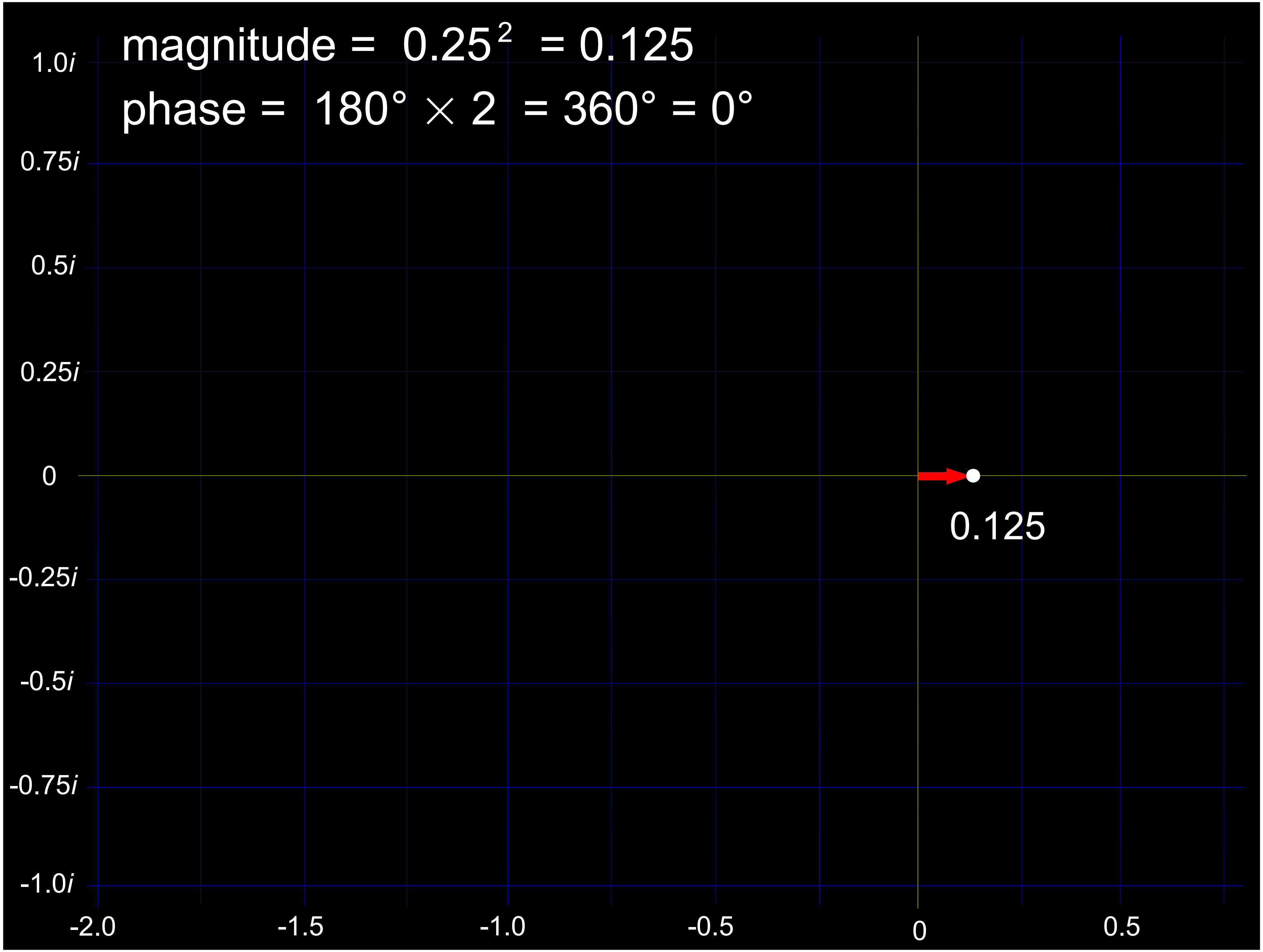
What is z^2 **squared**?



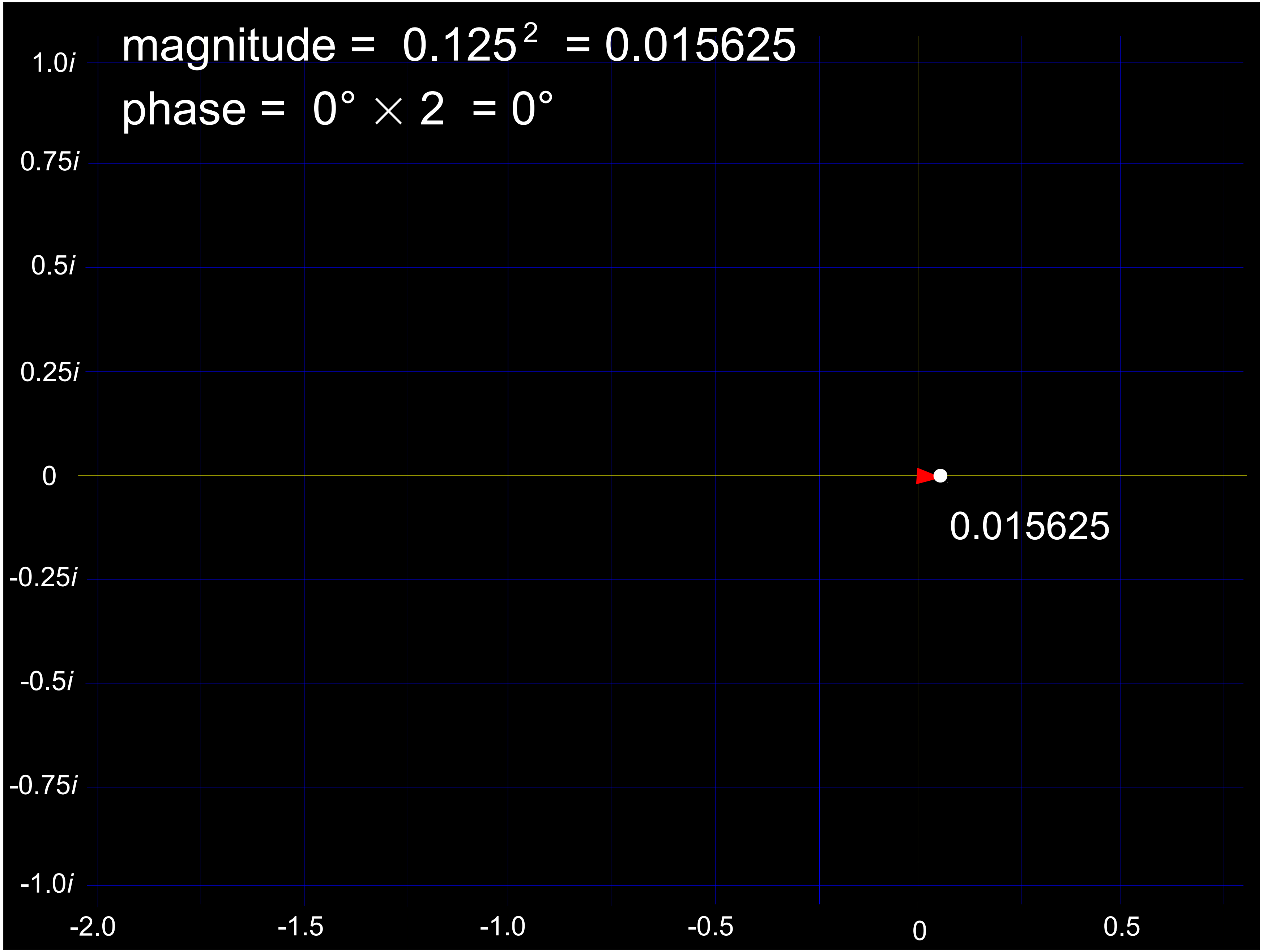
What is z^2 **squared**?



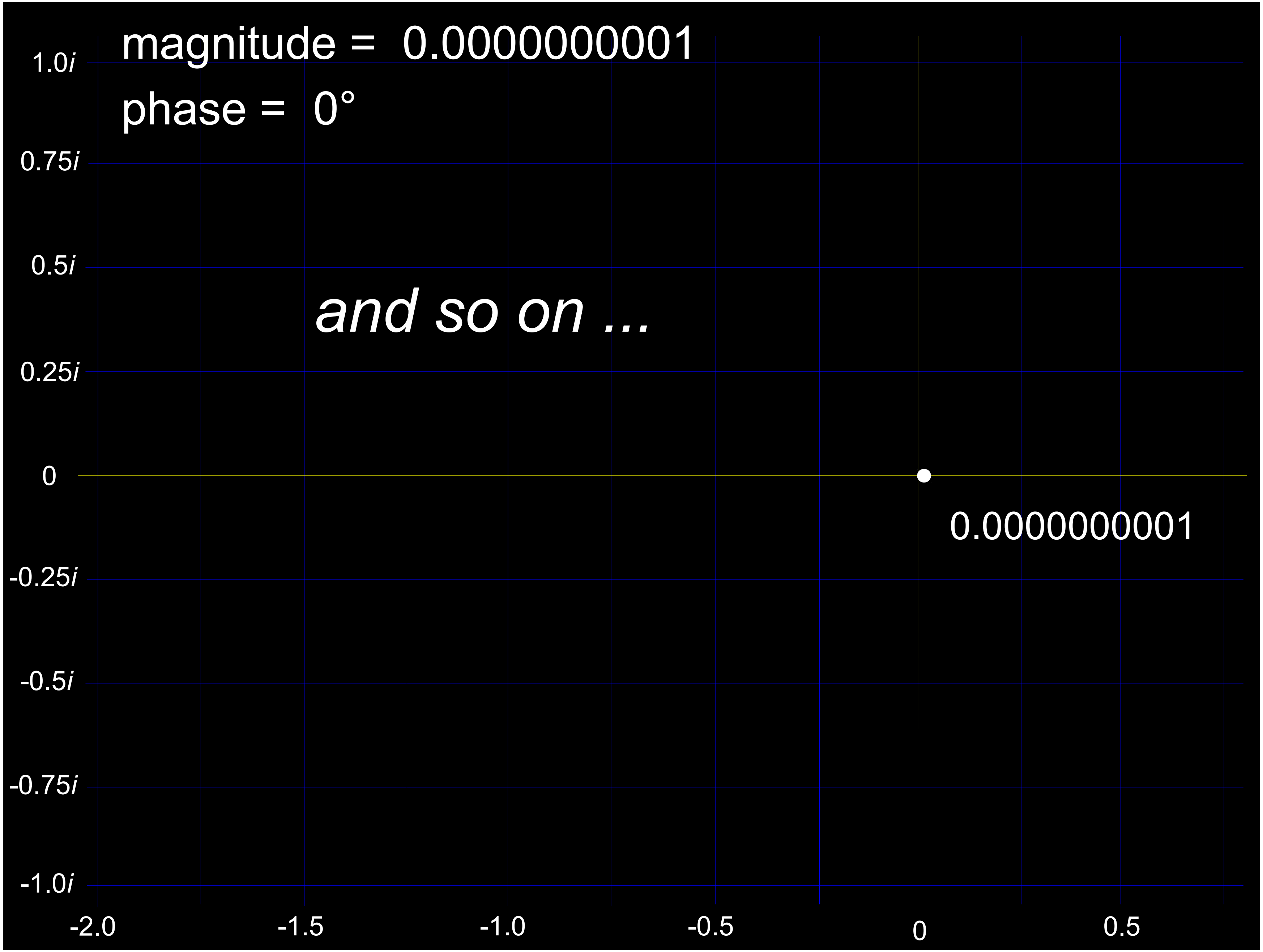
What If We Keep Going?



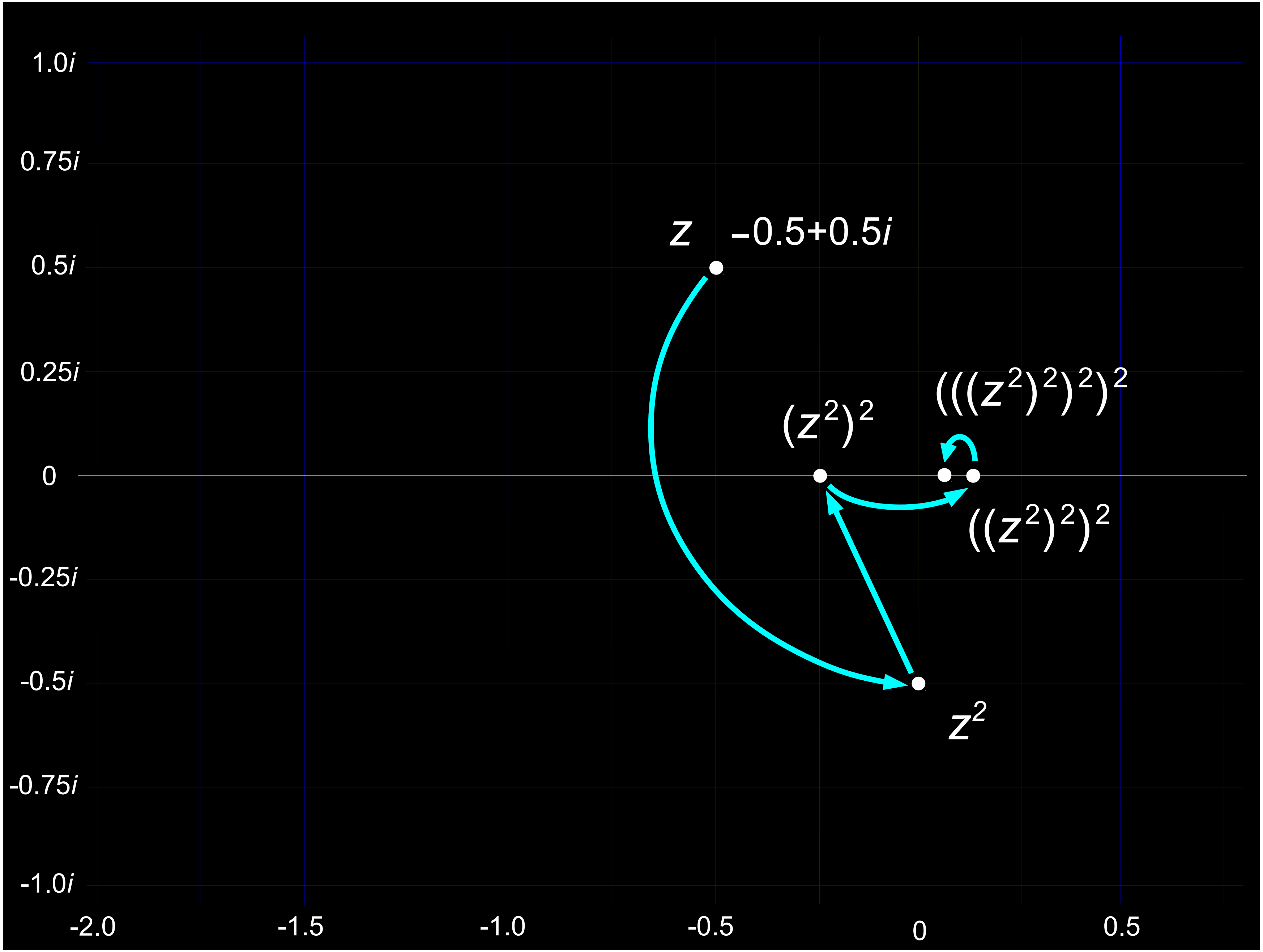
What If We Keep Going?



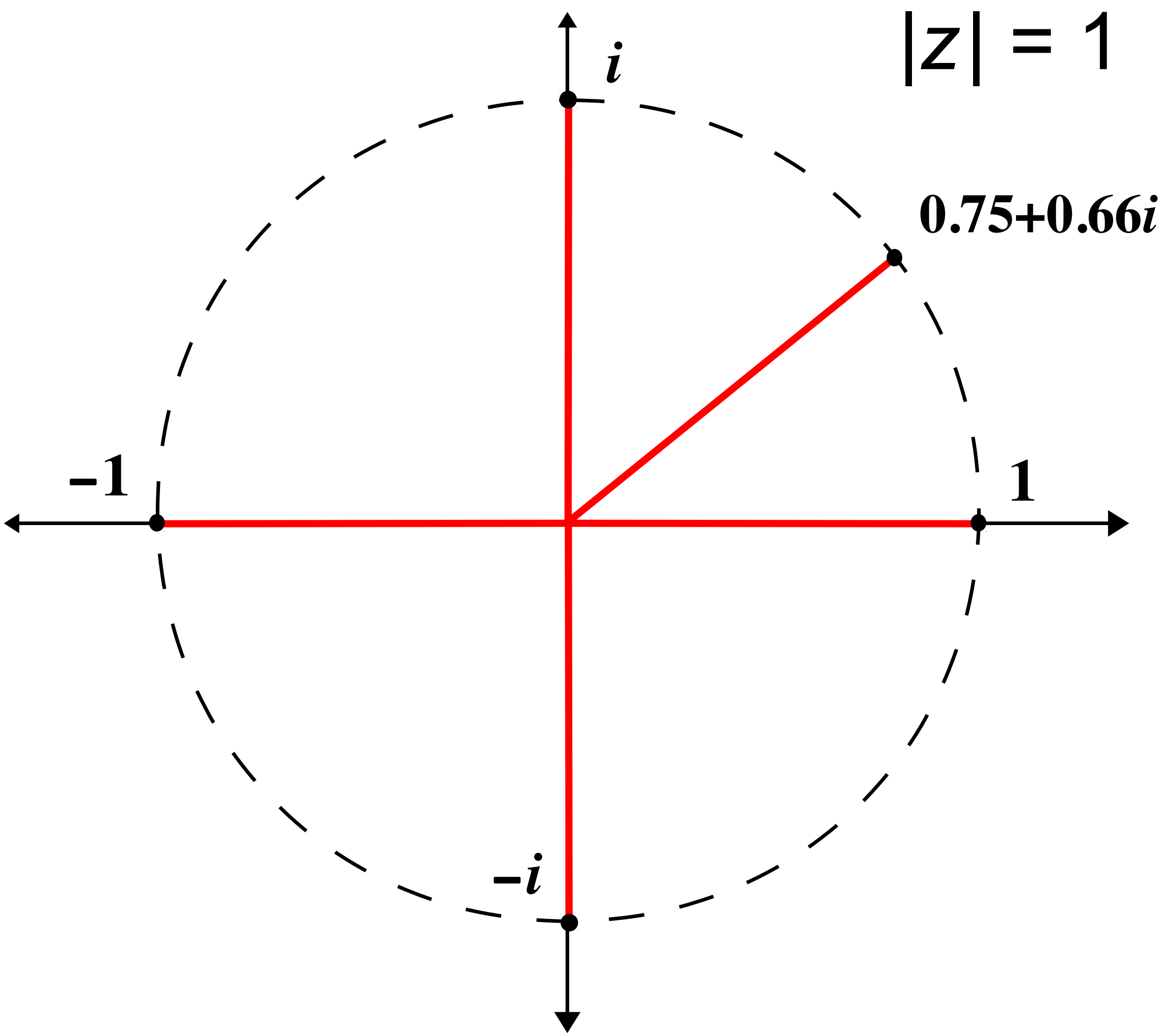
What If We Keep Going?



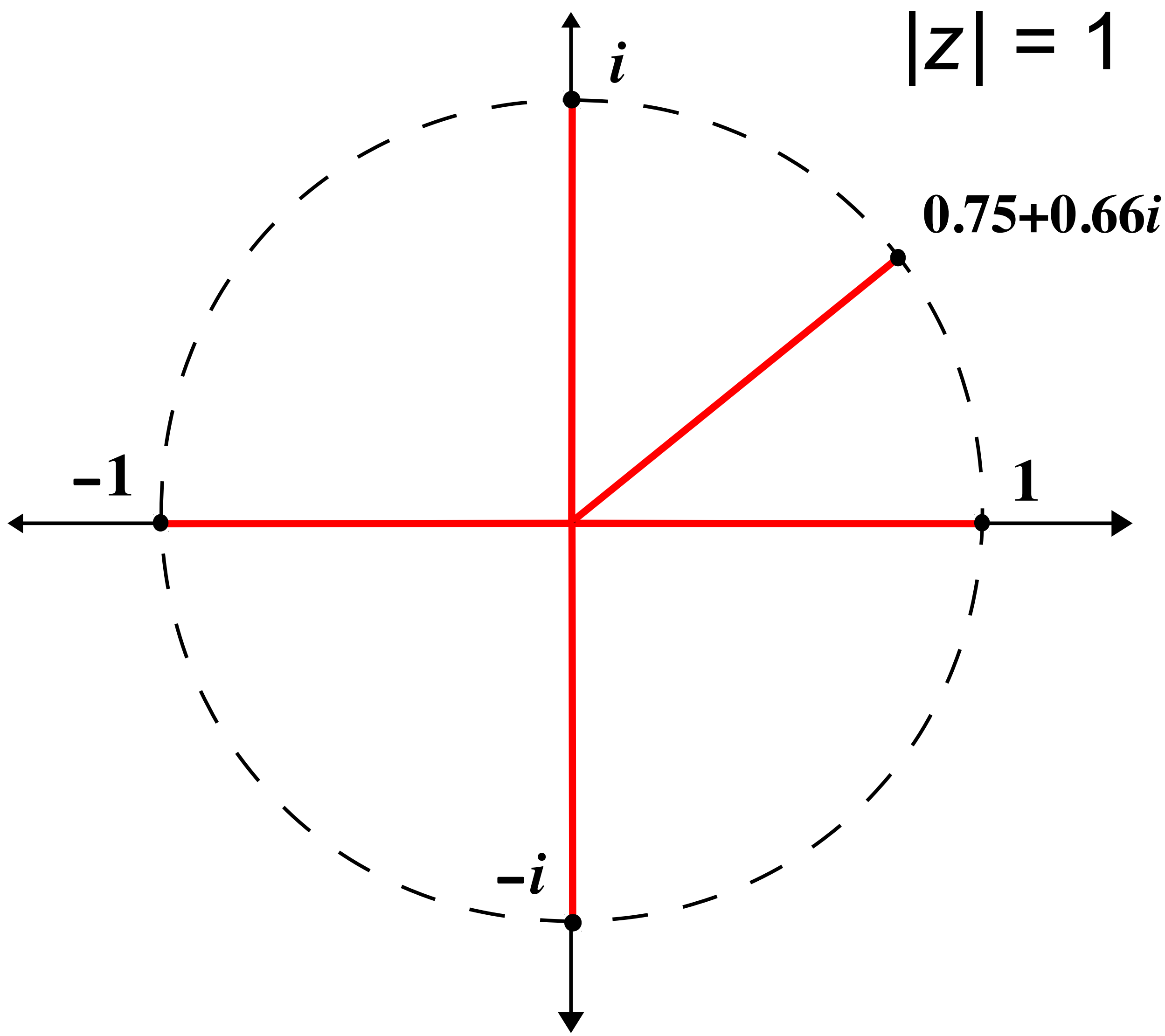
Notation: $z \rightarrow z^2$



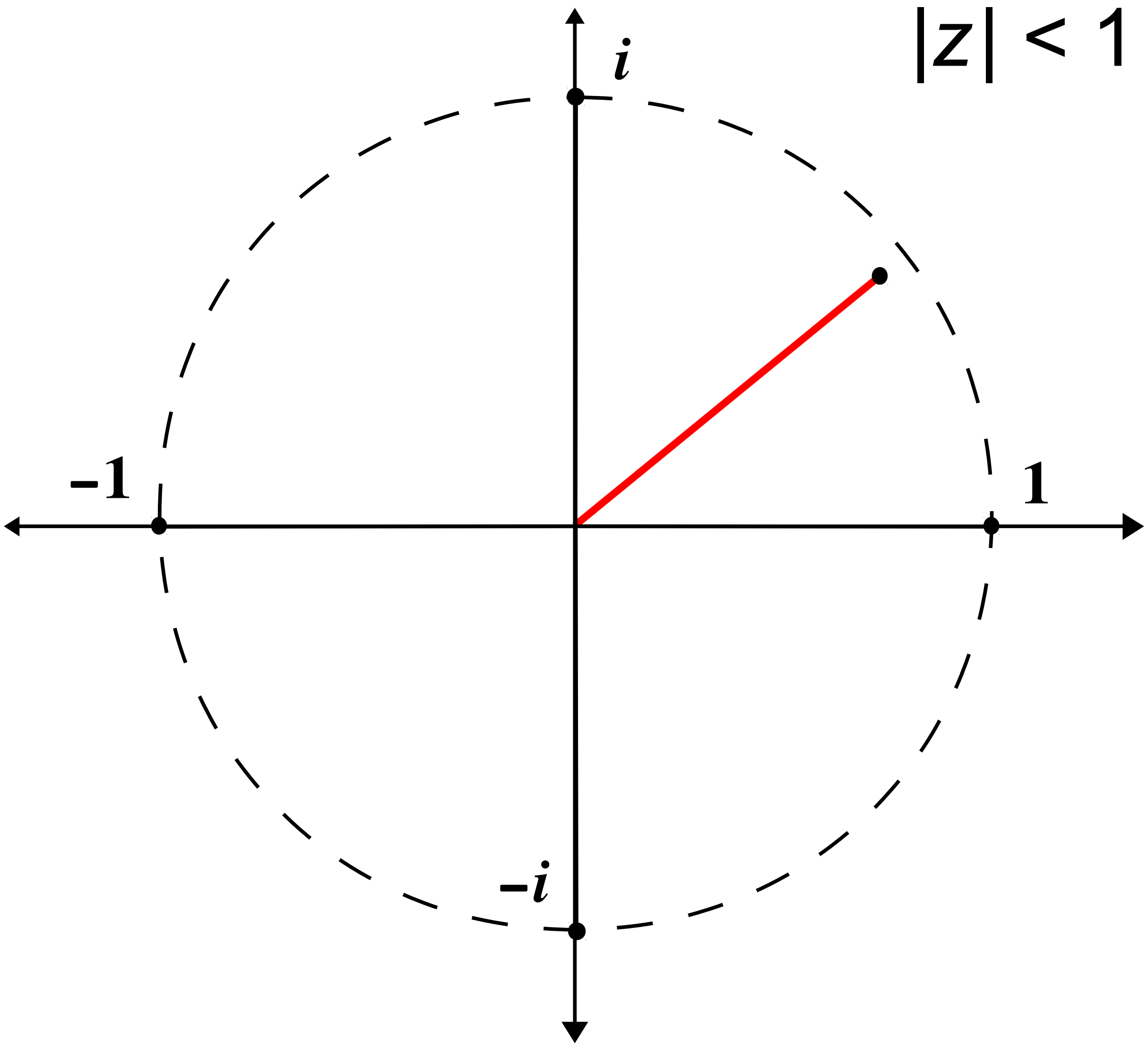
Numbers with Magnitude = 1



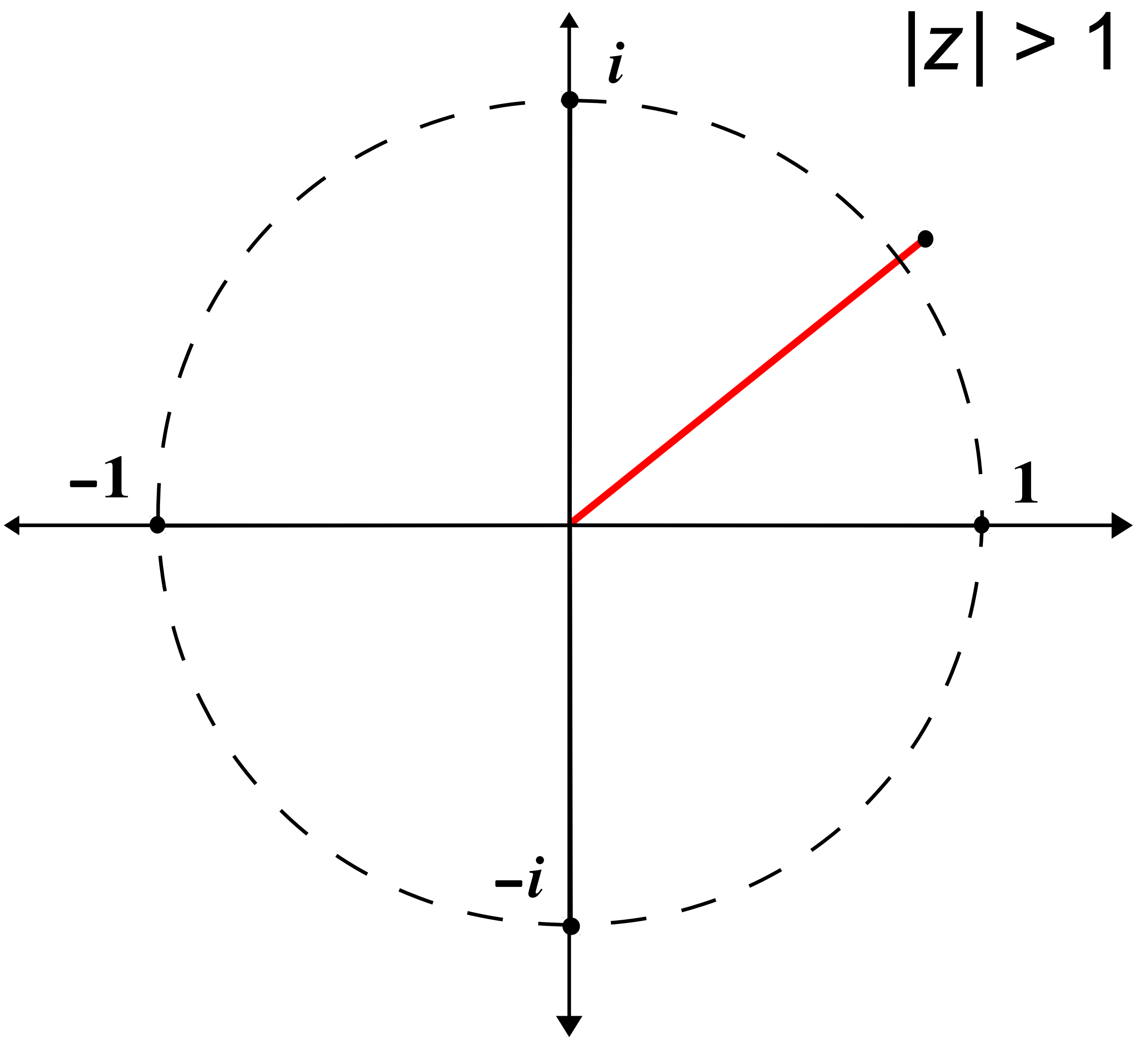
What Happens When We Repeatedly Square Numbers with Magnitude = 1 ?



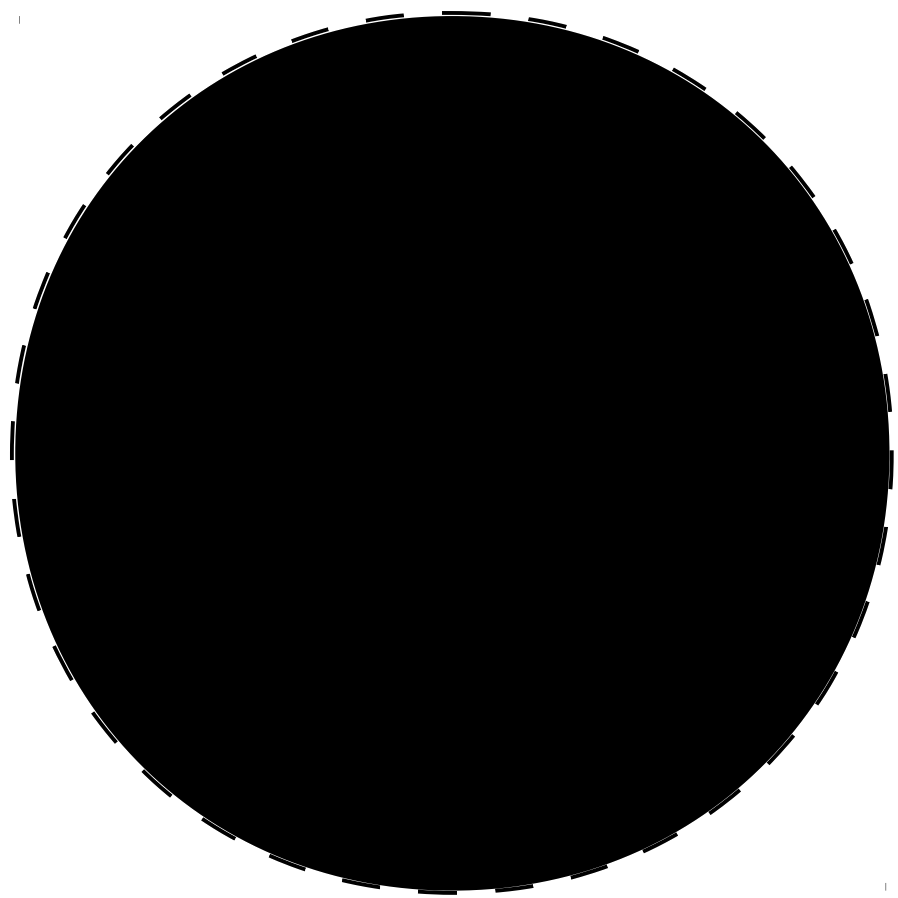
What Happens When We Repeatedly Square Numbers with Magnitude < 1 ?



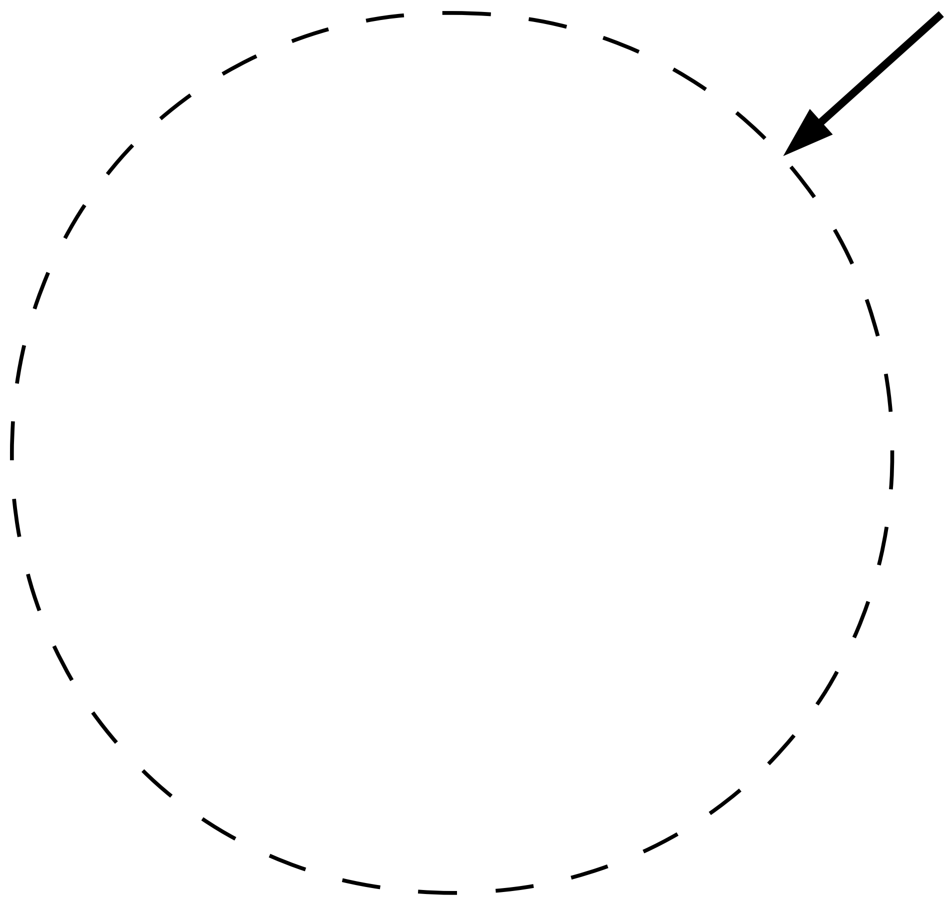
What Happens When We Repeatedly Square Numbers with Magnitude > 1 ?



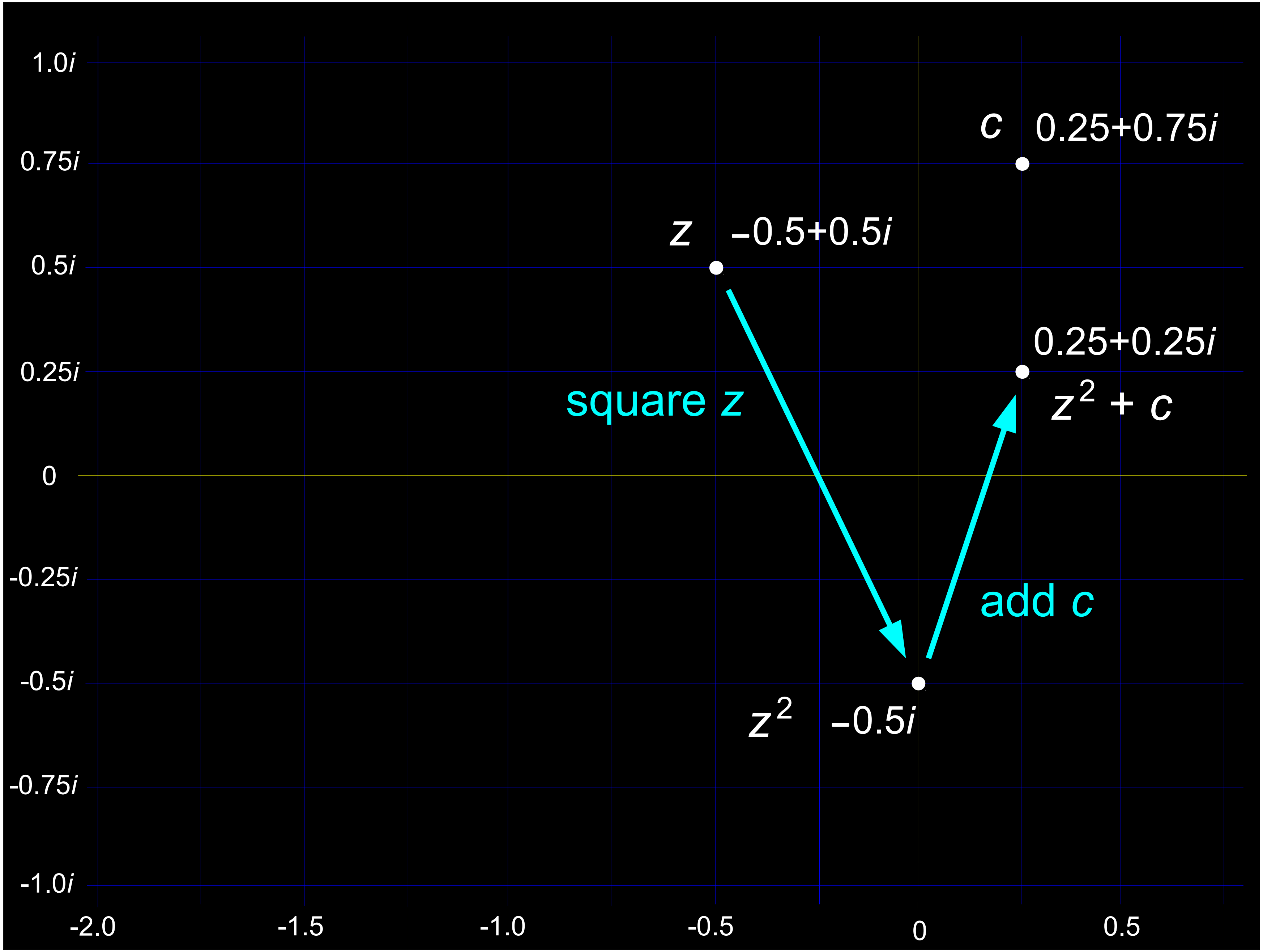
All Black Points “Converge” to Zero
All White Points “Diverge” to Infinity



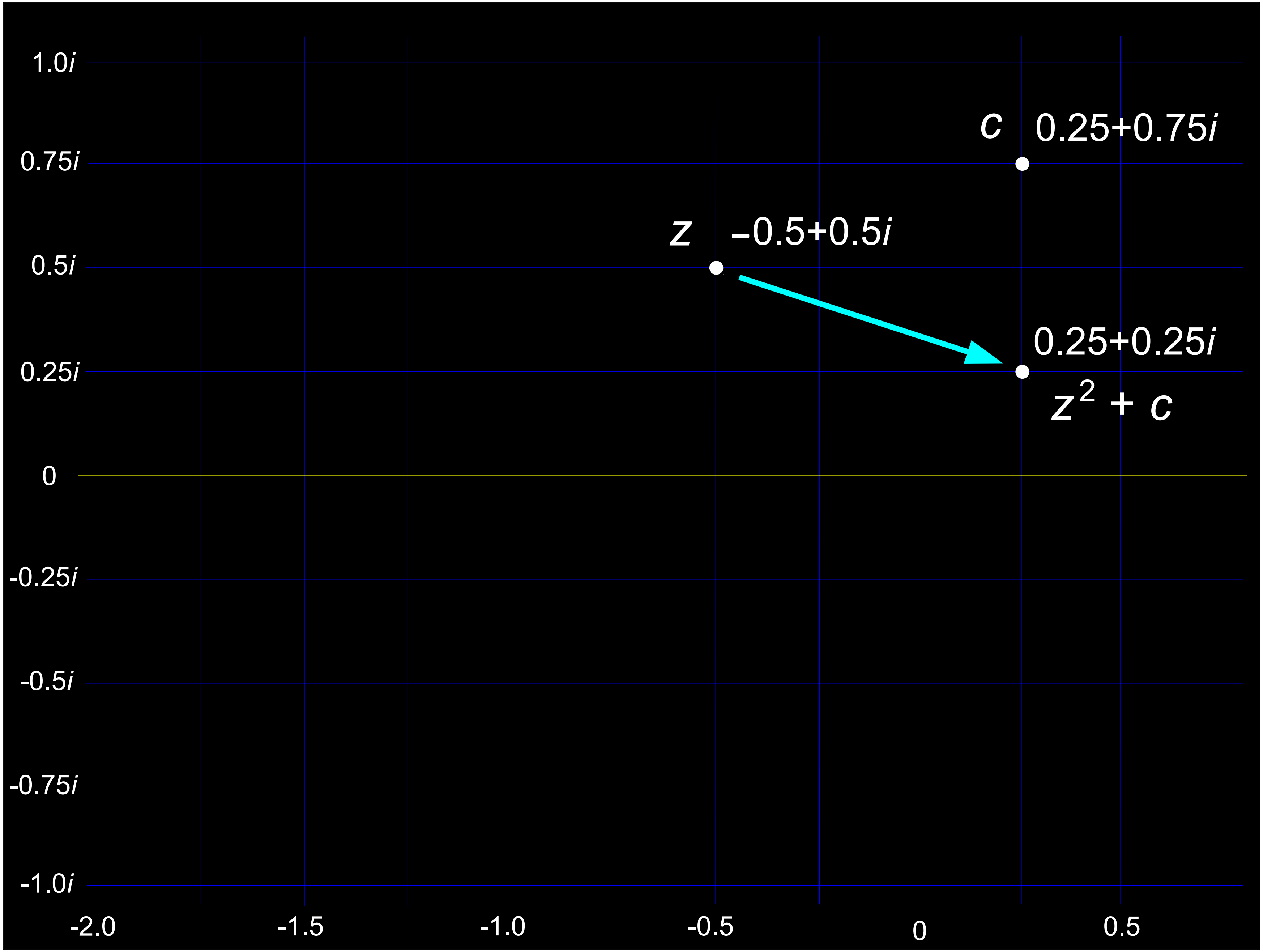
The Boundary Points Form a Perfect Circle



$z = -0.5+0.5i$ $c = 0.25+0.75i$



Notation: $z \rightarrow z^2 + c$



Now, choose any complex number c

What happens when we repeatedly apply

$$z \to z^2 + c$$

starting with c ?

$$c \;\to\; c^2 + c \;\to\; (c^2 + c)^2 + c \;\to\; ((c^2 + c)^2 + c)^2 + c$$

and so on ...

Now, choose any complex number c

What happens when we repeatedly apply

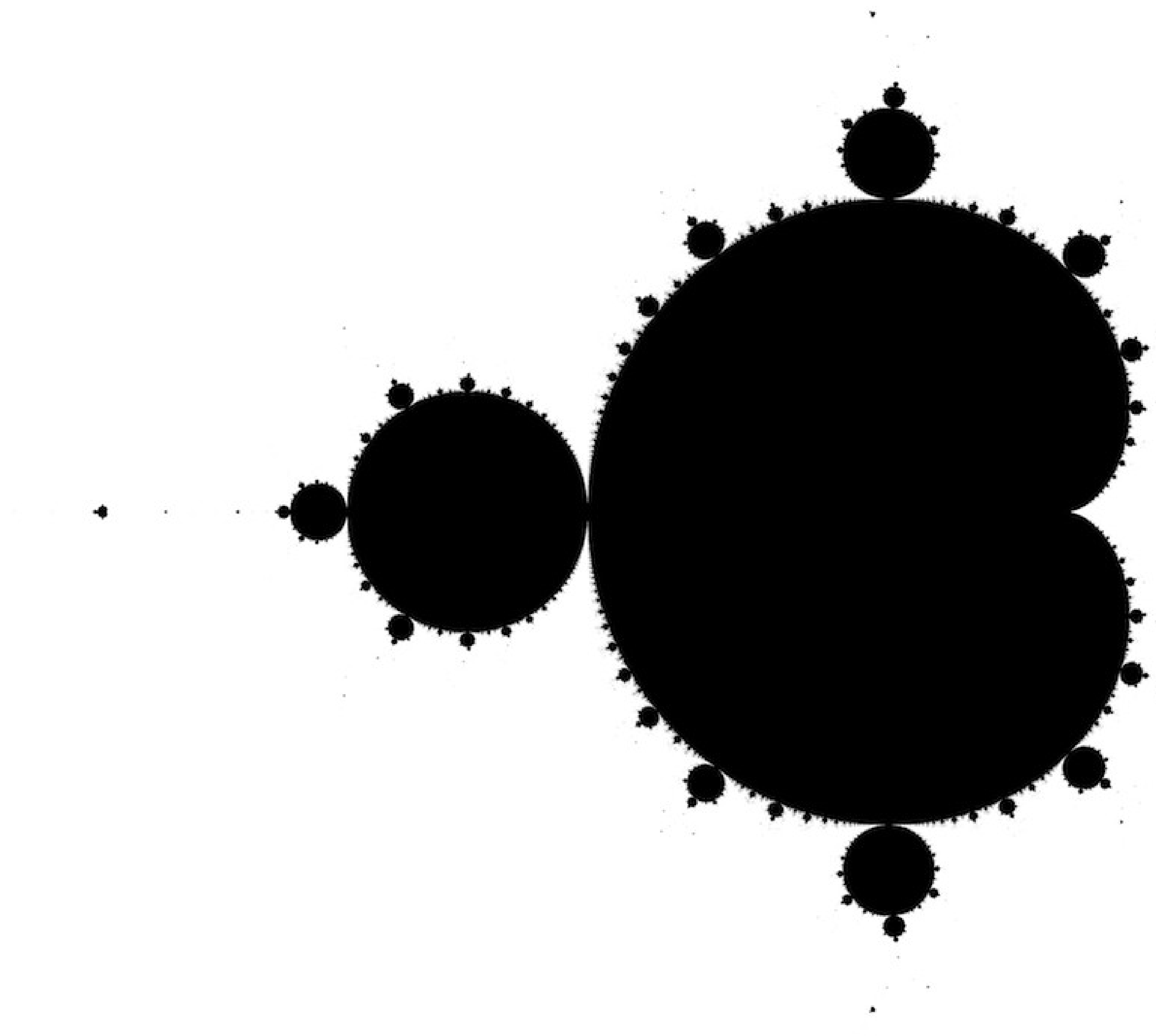
$$z \rightarrow z^2 + c$$

starting with **zero** ?

$$0 \;\rightarrow\; c \;\rightarrow\; c^2 + c \;\rightarrow\; (c^2 + c)^2 + c \;\rightarrow\; \ldots$$

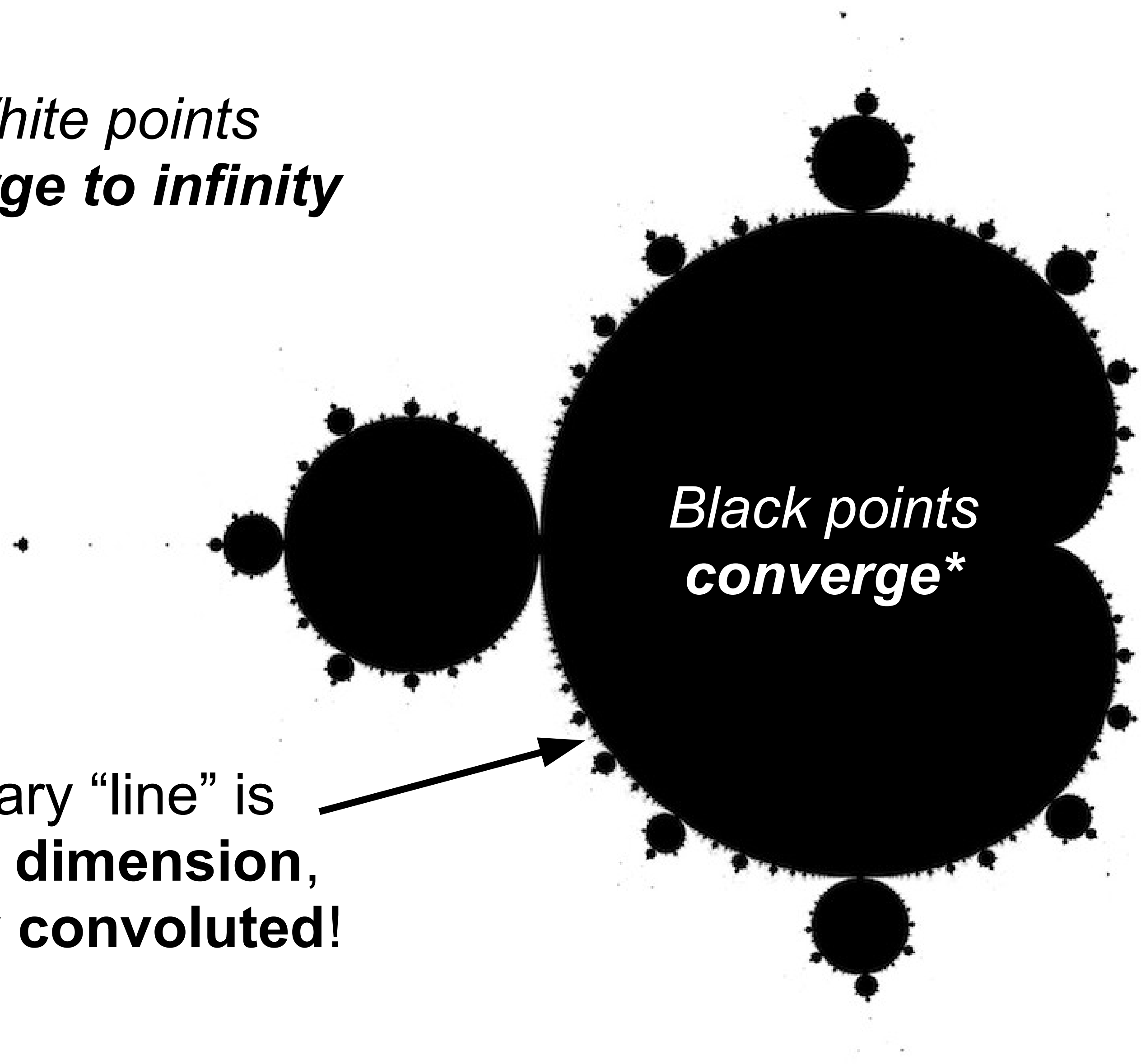
The process is the same

The Mandelbrot Set



The Mandelbrot Set

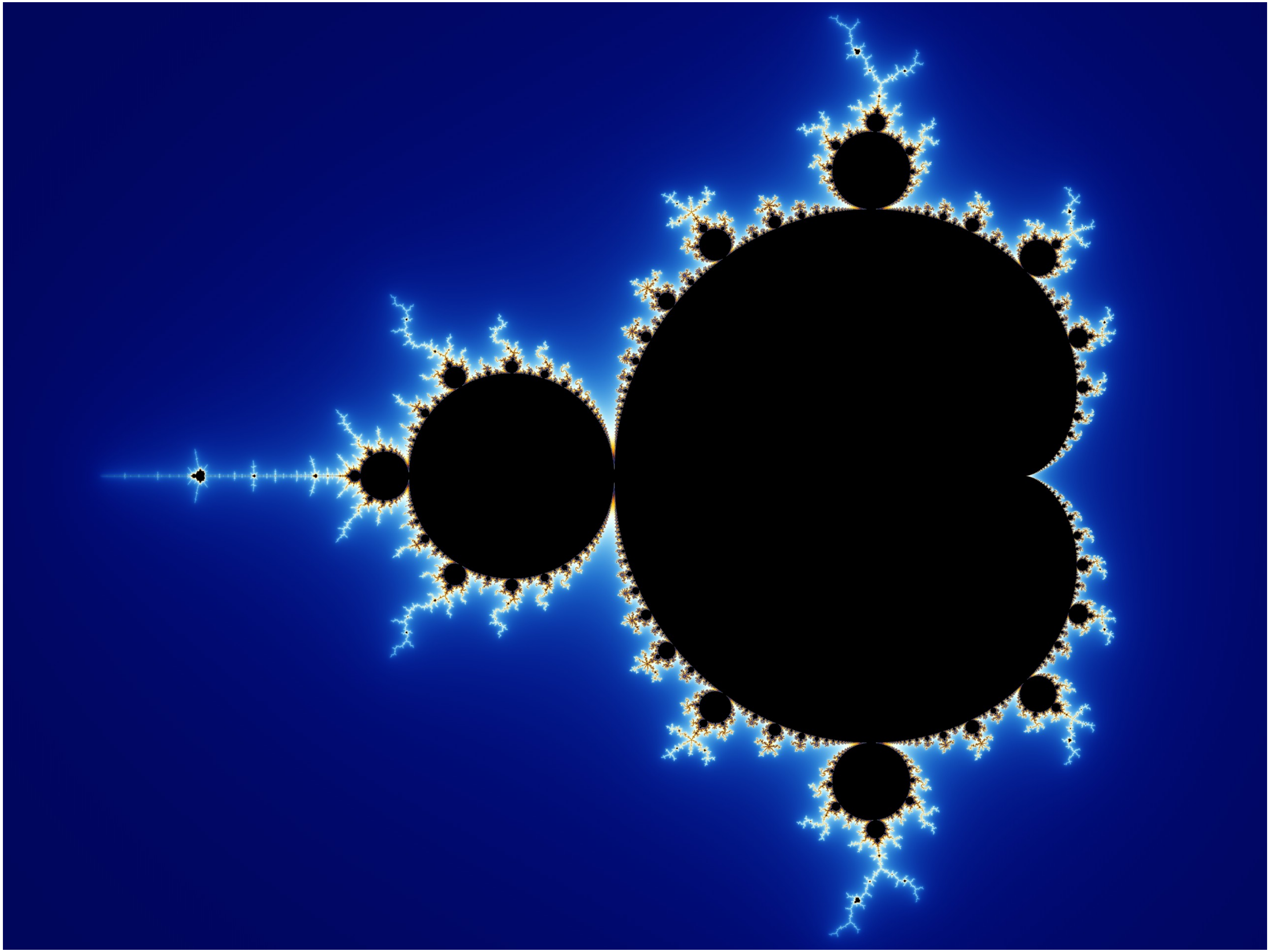
White points
diverge to infinity



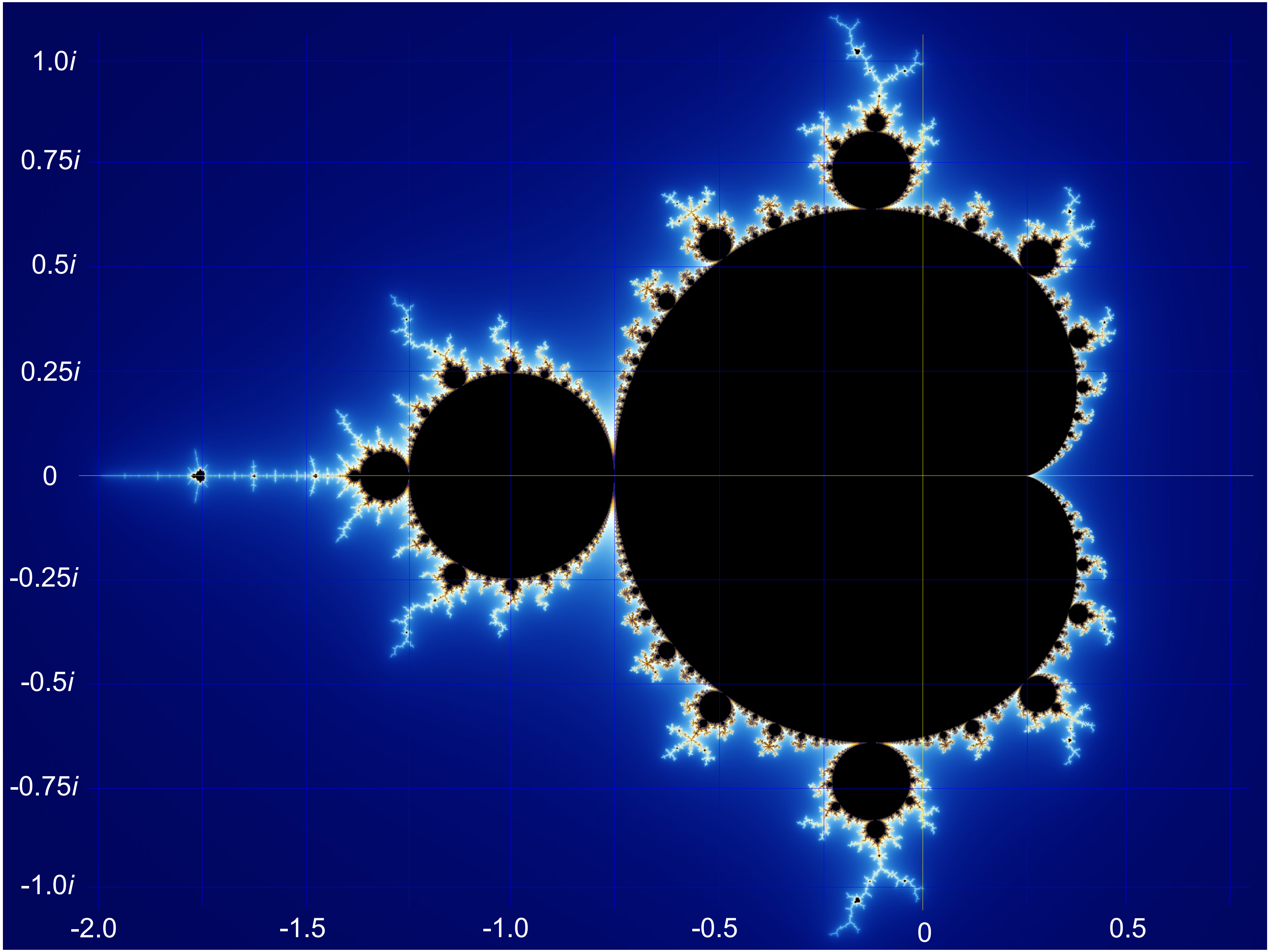
The boundary "line" is
of **fractional dimension**,
and **infinitely convoluted**!

* but not necessarily to zero, or even to a single, fixed point

The Mandelbrot Set



The Mandelbrot Set

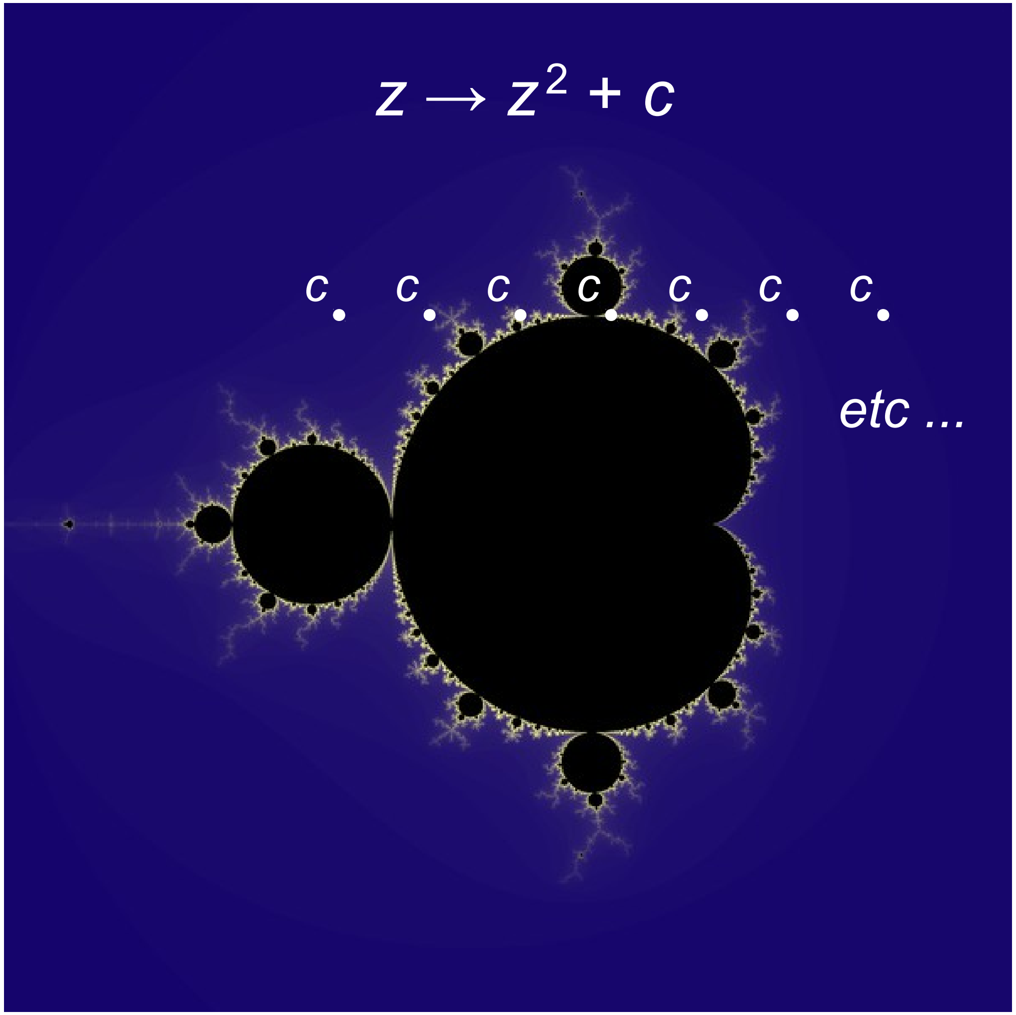


How to Color a Pixel

Let c be a complex number that corresponds to the pixel

- Initialize $z = 0$
- Repeatedly apply the update rule: $z \rightarrow z^2 + c$
- See how long it takes for the magnitude of z to exceed 2
  - If z's magnitude never exceeds 2, color the pixel **black**
  - Otherwise, choose a color based on **how many steps** it took for z's magnitude to exceed 2

The Mandelbrot Set Algorithm



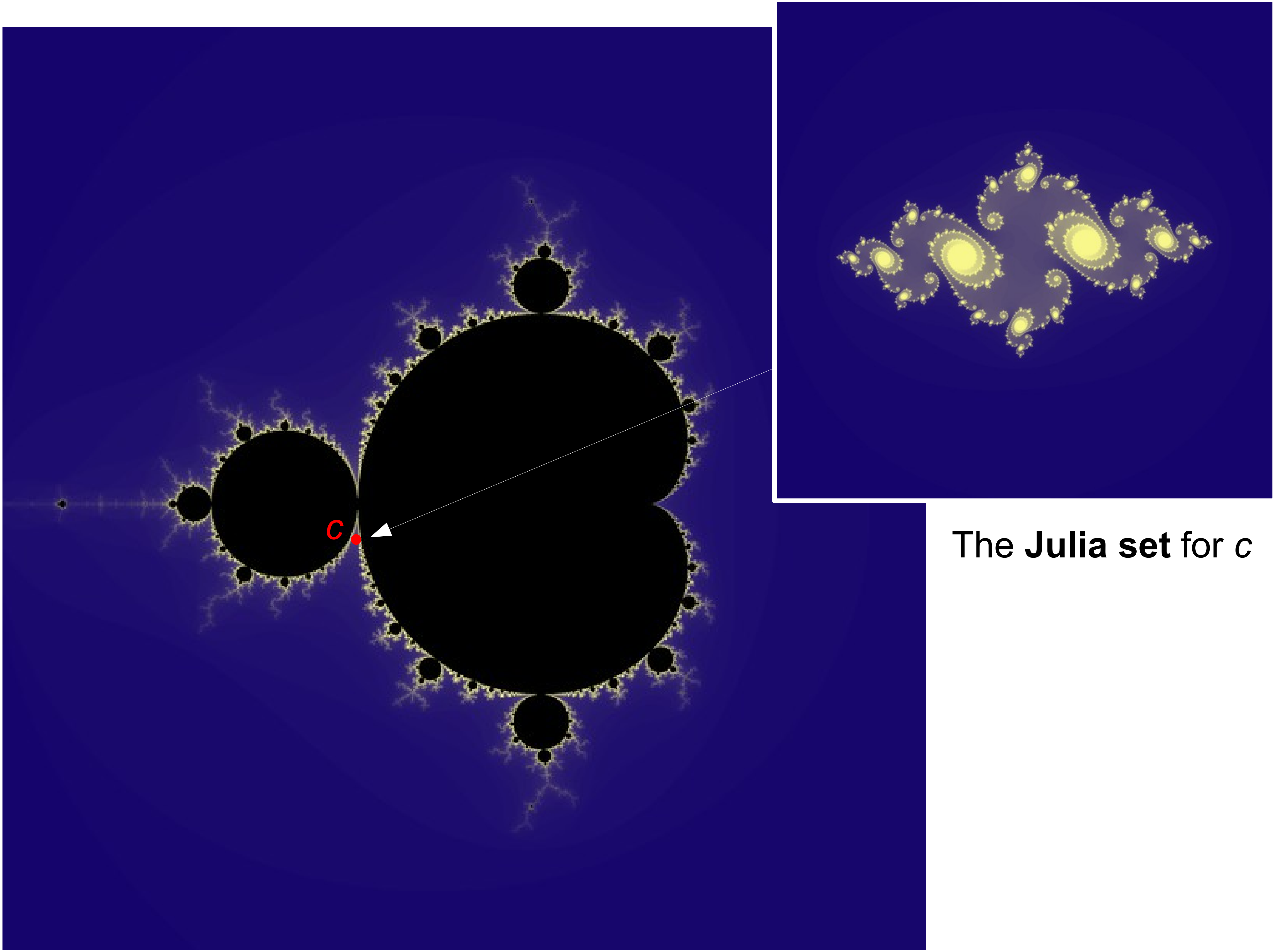
We use different values for *c*, and always start the iteration at 0

The Mandelbrot Set Algorithm

- For every number c in the complex plane, do the following:

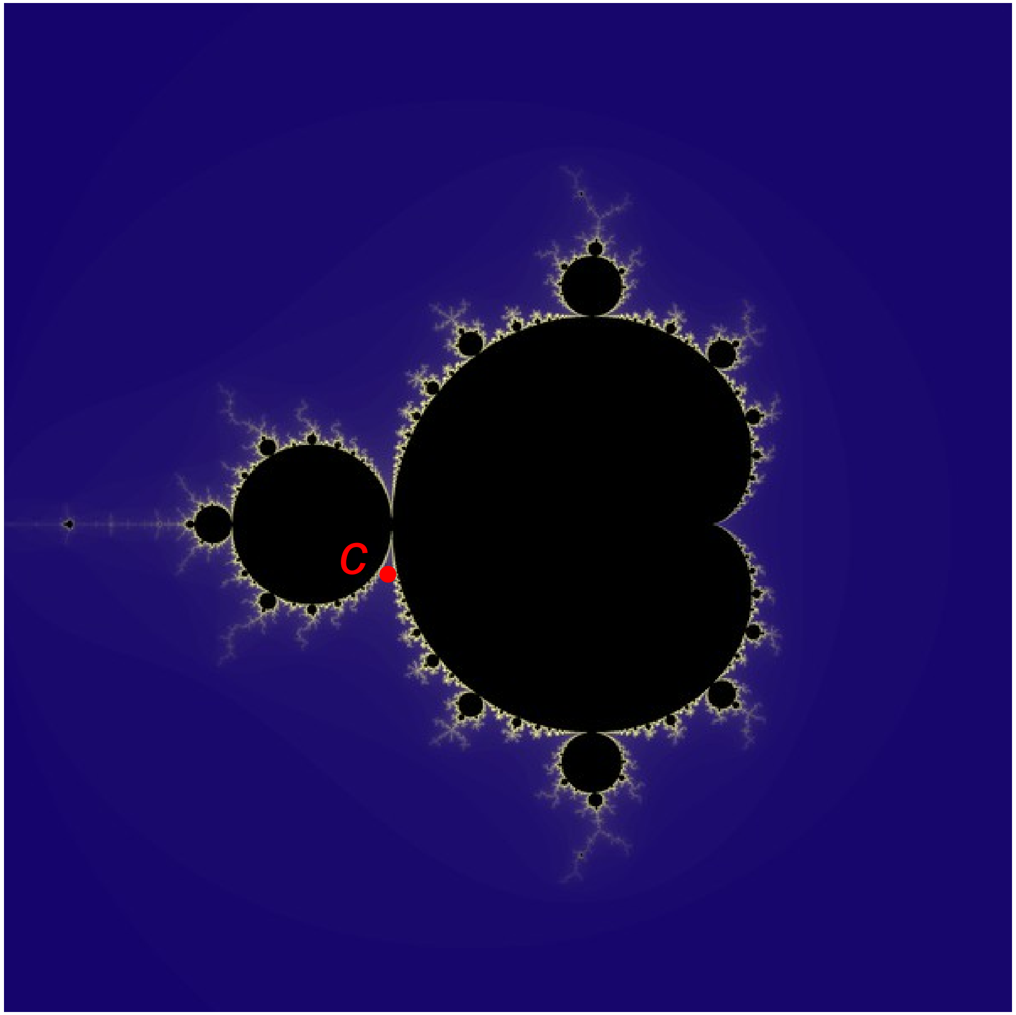
  1. Initialize z to 0
  2. Initialize *count* to 0
  3. If the magnitude of $z > 2$, choose a color for c based on the value of *count*, and stop; otherwise continue
  4. Increase *count* by 1
  5. Compute $z^2 + c$ and make this the new value of z
  6. Go to step 3

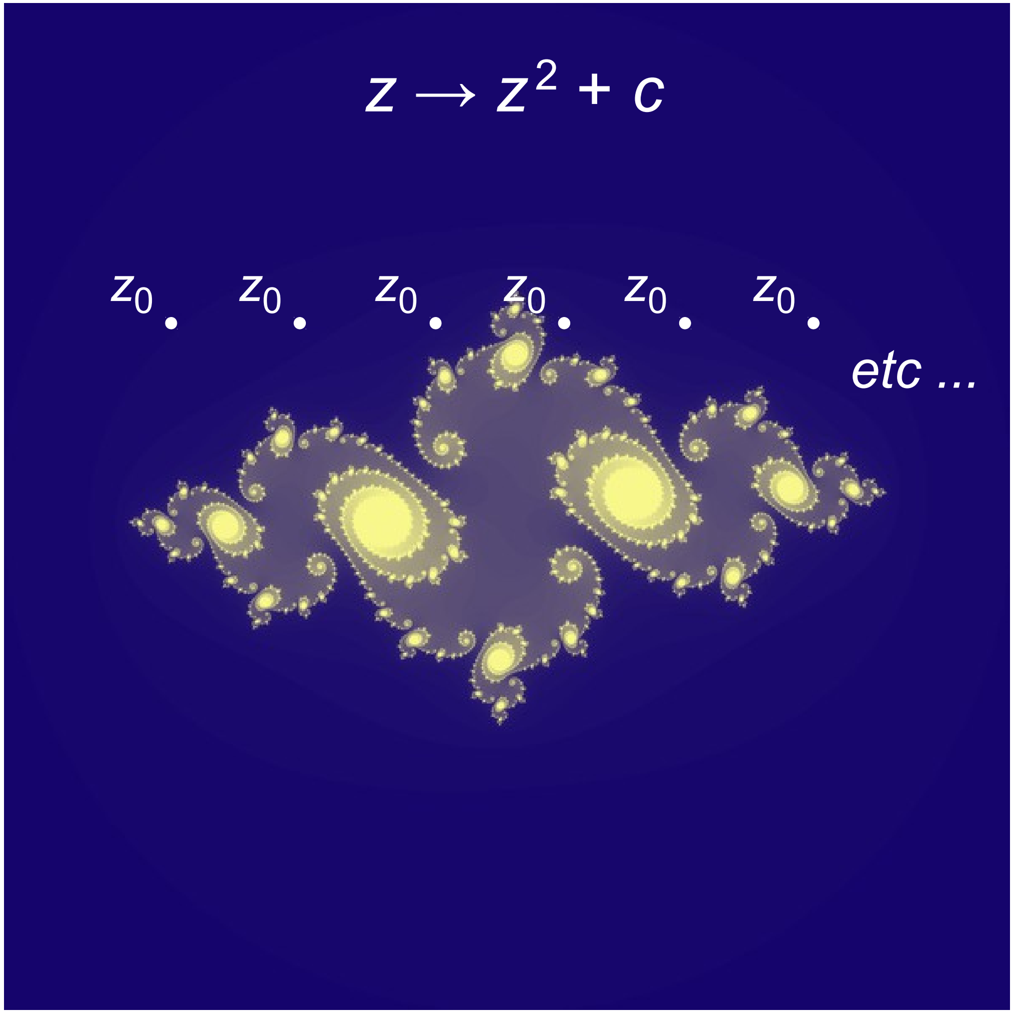
  If the loop never stops, color c black



The **Julia set** for *c*

The Julia Set Algorithm for the Number c





We keep the value of c fixed, and start the iteration at different values of z_0

The Julia Set Algorithm for the Number *c*

- For every number z_0 in the complex plane, do the following:

  1. Initialize z to z_0
  2. Initialize *count* to 0
  3. If the magnitude of $z > 2$, choose a color for z_0 based on the value of *count*, and stop; otherwise continue
  4. Increase *count* by 1
  5. Compute $z^2 + c$ and make this the new value of z
  6. Go to step 3

  If the loop never stops, color z_0 black