# “Mathematical Monsters”

# The Idea of a Limit

- What is the value of the infinite sum:

$$1/2 + 1/4 + 1/8 + 1/16 + 1/32 + 1/64 + 1/128 + \ldots$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = ?$$

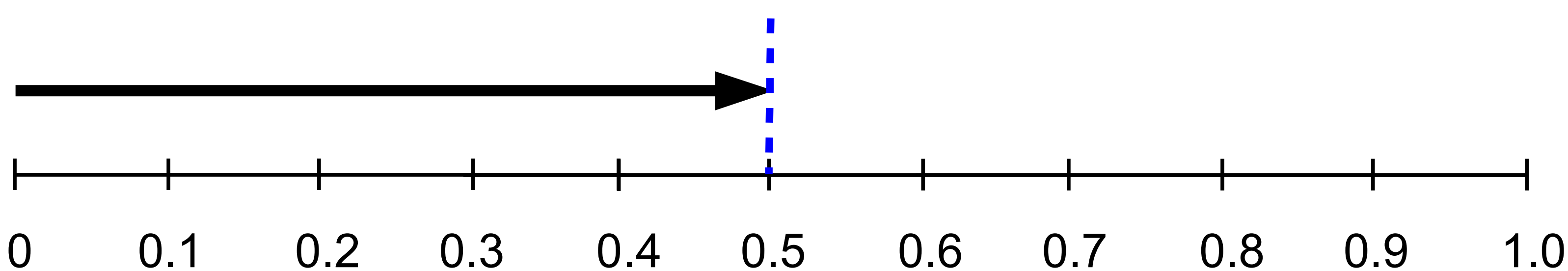
# The Idea of a Limit

- What is the value of the infinite sum:

$$1/2 + 1/4 + 1/8 + 1/16 + 1/32 + 1/64 + 1/128 + \ldots$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = ?$$

$$1/2 = 0.5$$



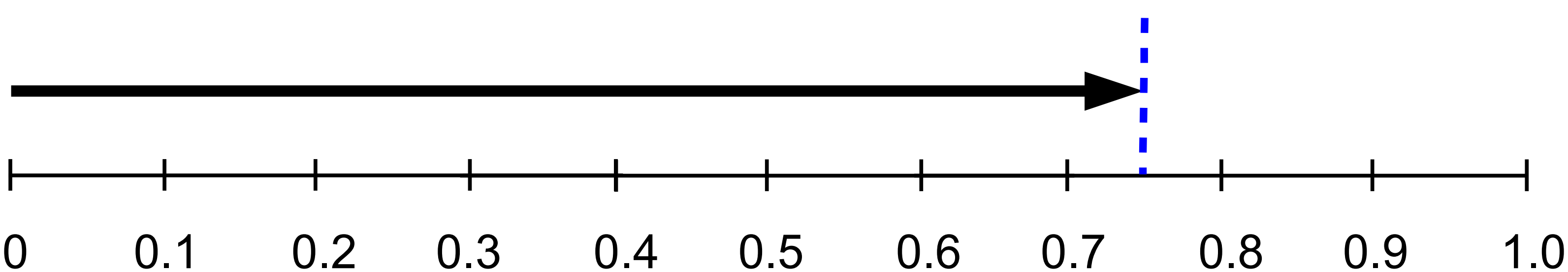
# The Idea of a Limit

- What is the value of the infinite sum:

$$1/2 + 1/4 + 1/8 + 1/16 + 1/32 + 1/64 + 1/128 + \ldots$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = ?$$

$1/2 + 1/4 = 0.75$



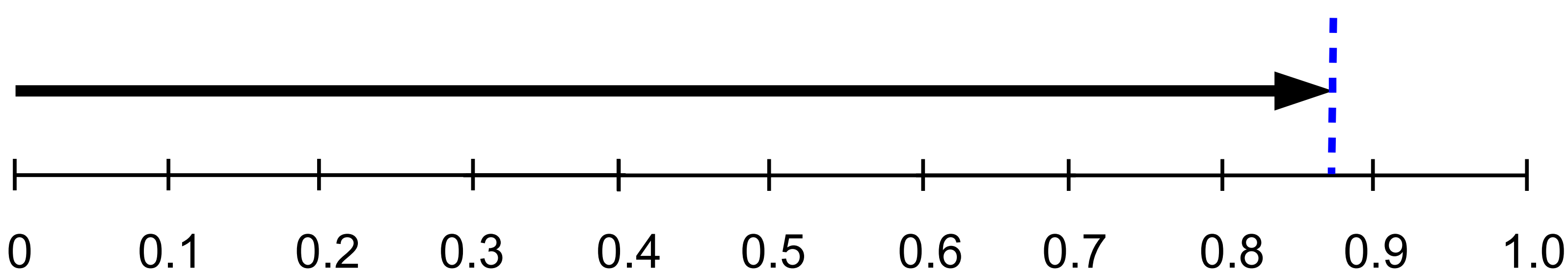
# The Idea of a Limit

- What is the value of the infinite sum:

$$1/2 + 1/4 + 1/8 + 1/16 + 1/32 + 1/64 + 1/128 + \ldots$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = ?$$

$$1/2 + 1/4 \color{blue}{+ 1/8 = 0.875}$$



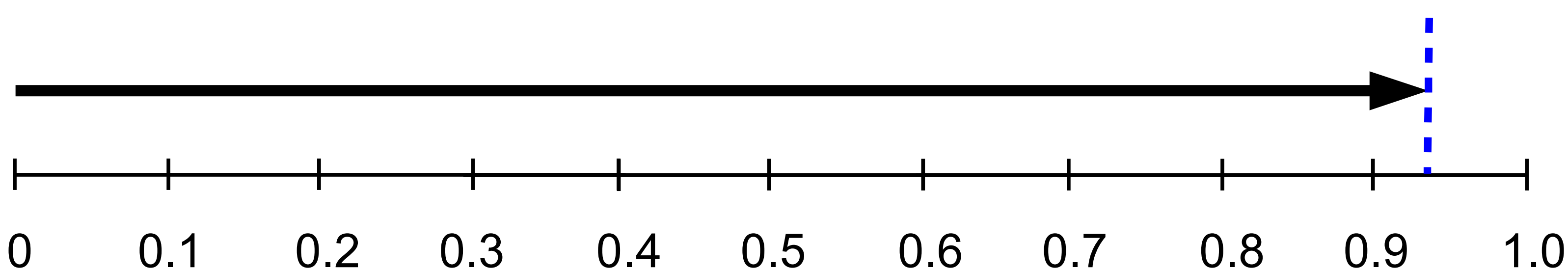
# The Idea of a Limit

- What is the value of the infinite sum:

$$1/2 + 1/4 + 1/8 + 1/16 + 1/32 + 1/64 + 1/128 + \ldots$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = ?$$

$$1/2 + 1/4 + 1/8 + 1/16 = 0.9375$$



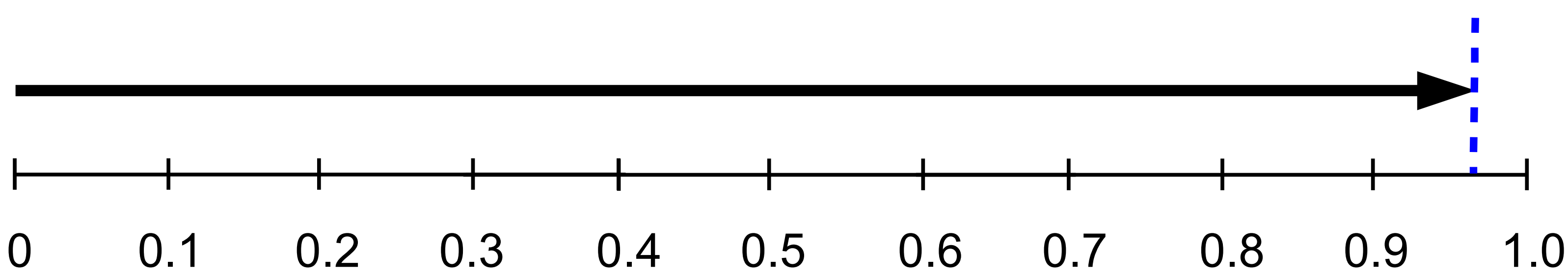
# The Idea of a Limit

- What is the value of the infinite sum:

$$1/2 + 1/4 + 1/8 + 1/16 + 1/32 + 1/64 + 1/128 + \ldots$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = ?$$

$$1/2 + 1/4 + 1/8 + 1/16 + 1/32 = 0.96875$$



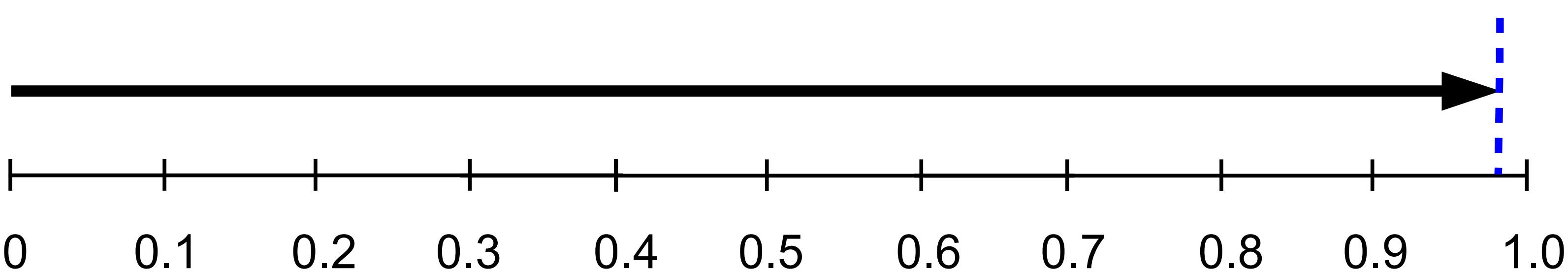
# The Idea of a Limit

- What is the value of the infinite sum:

$$1/2 + 1/4 + 1/8 + 1/16 + 1/32 + 1/64 + 1/128 + \ldots$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = ?$$

$$1/2 + 1/4 + 1/8 + 1/16 + 1/32 + 1/64 = 0.984375$$



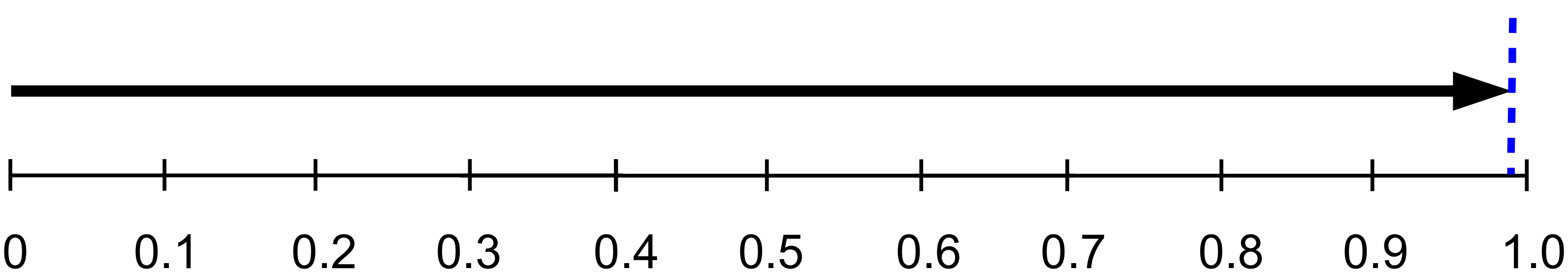
# The Idea of a Limit

- What is the value of the infinite sum:

$$1/2 + 1/4 + 1/8 + 1/16 + 1/32 + 1/64 + 1/128 + \ldots$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = ?$$

$$1/2 + 1/4 + 1/8 + 1/16 + 1/32 + 1/64 \color{blue}{+ 1/128 = 0.9921875}$$



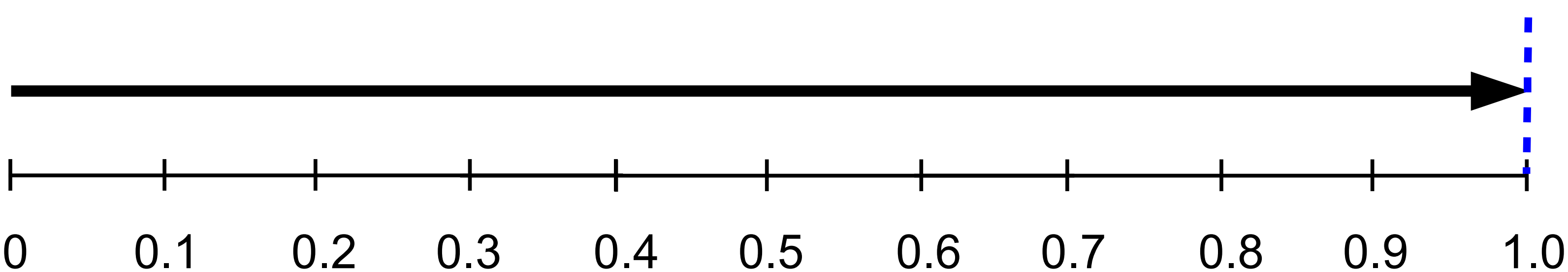
# The Idea of a Limit

- What is the value of the infinite sum:

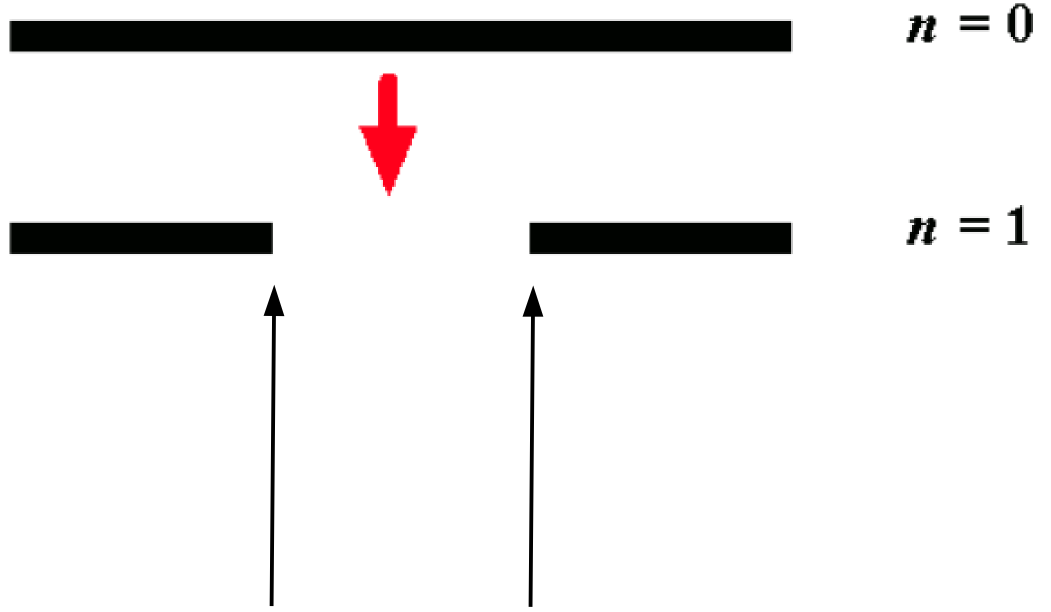
$$1/2 + 1/4 + 1/8 + 1/16 + 1/32 + 1/64 + 1/128 + \ldots$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \mathbf{1.0000000}$$

$$1/2 + 1/4 + 1/8 + 1/16 + 1/32 + 1/64 + 1/128 + \ldots$$

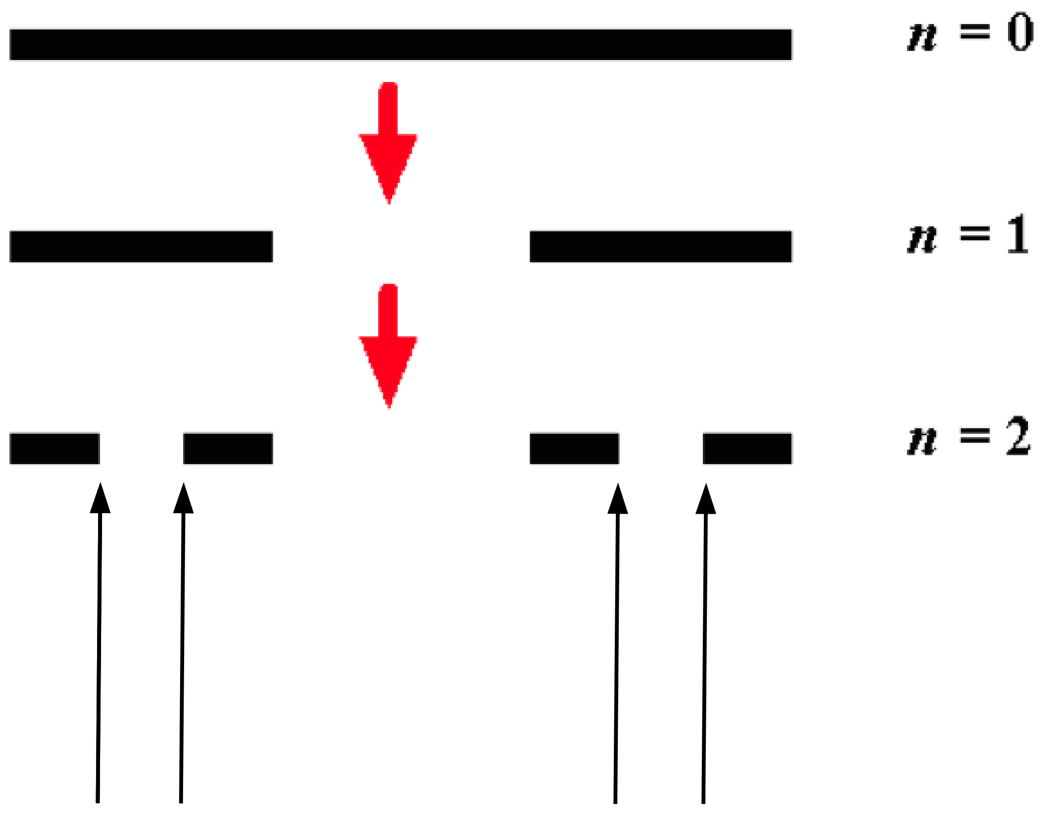


# The Cantor Set



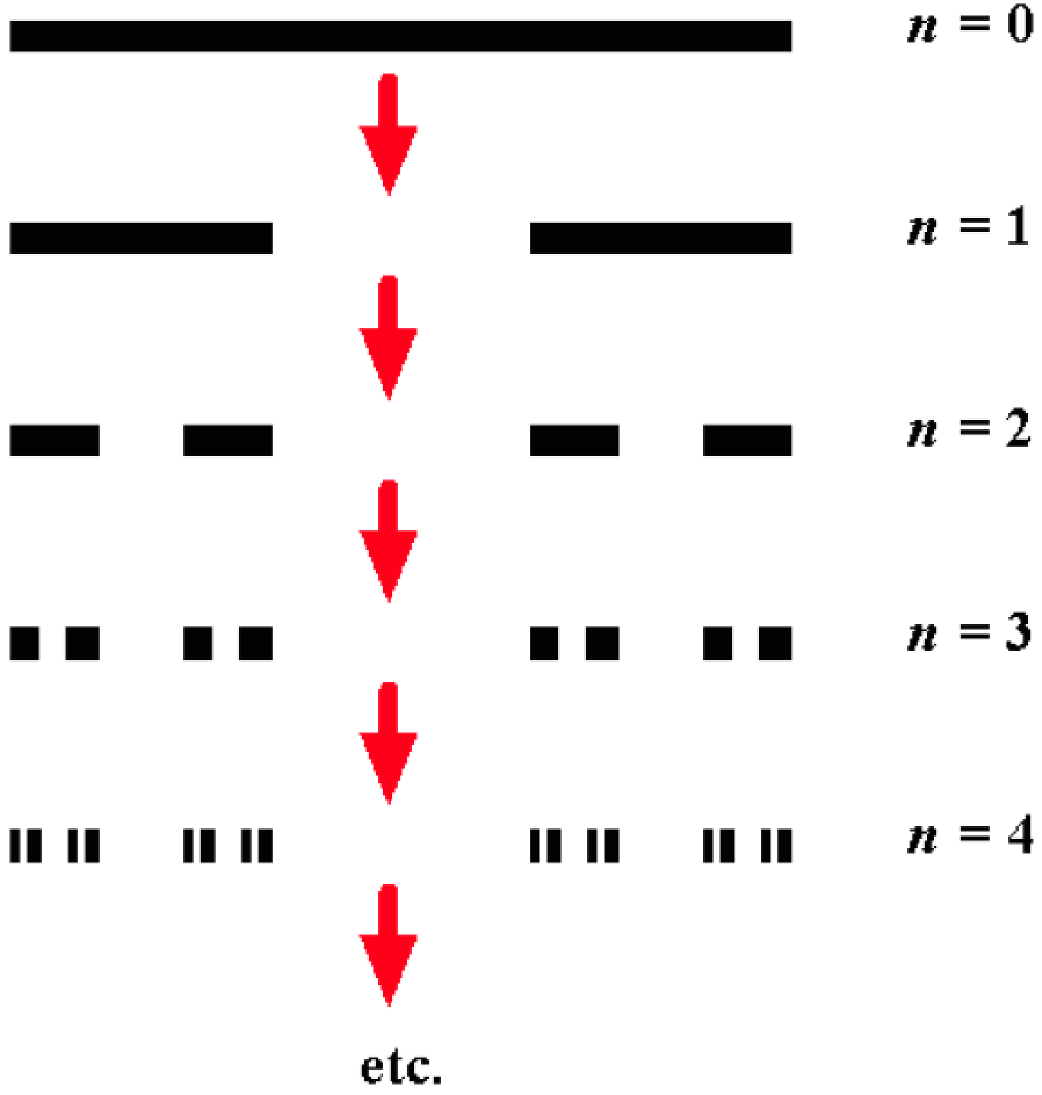

These endpoints are NOT removed

# The Cantor Set

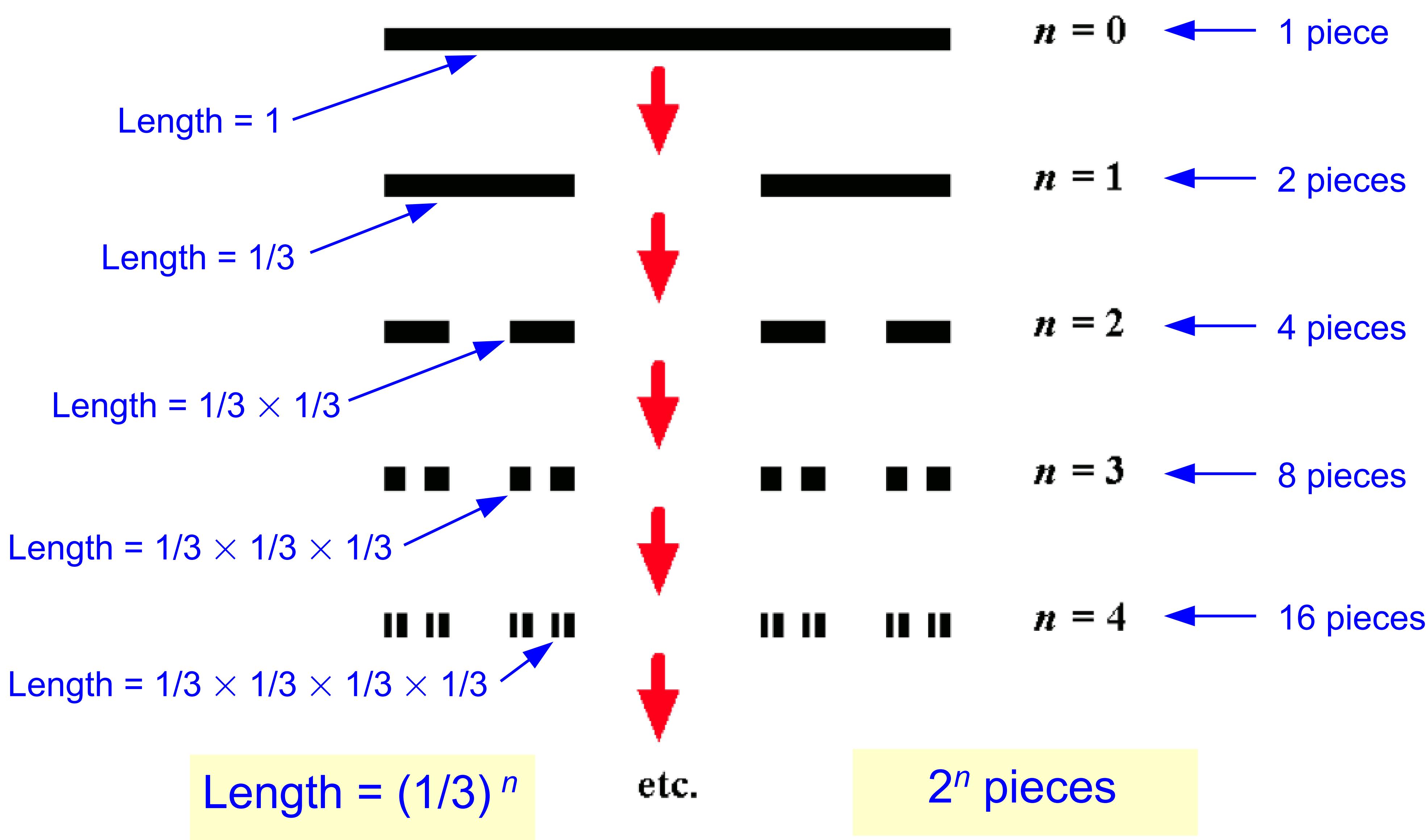



These endpoints are NOT removed

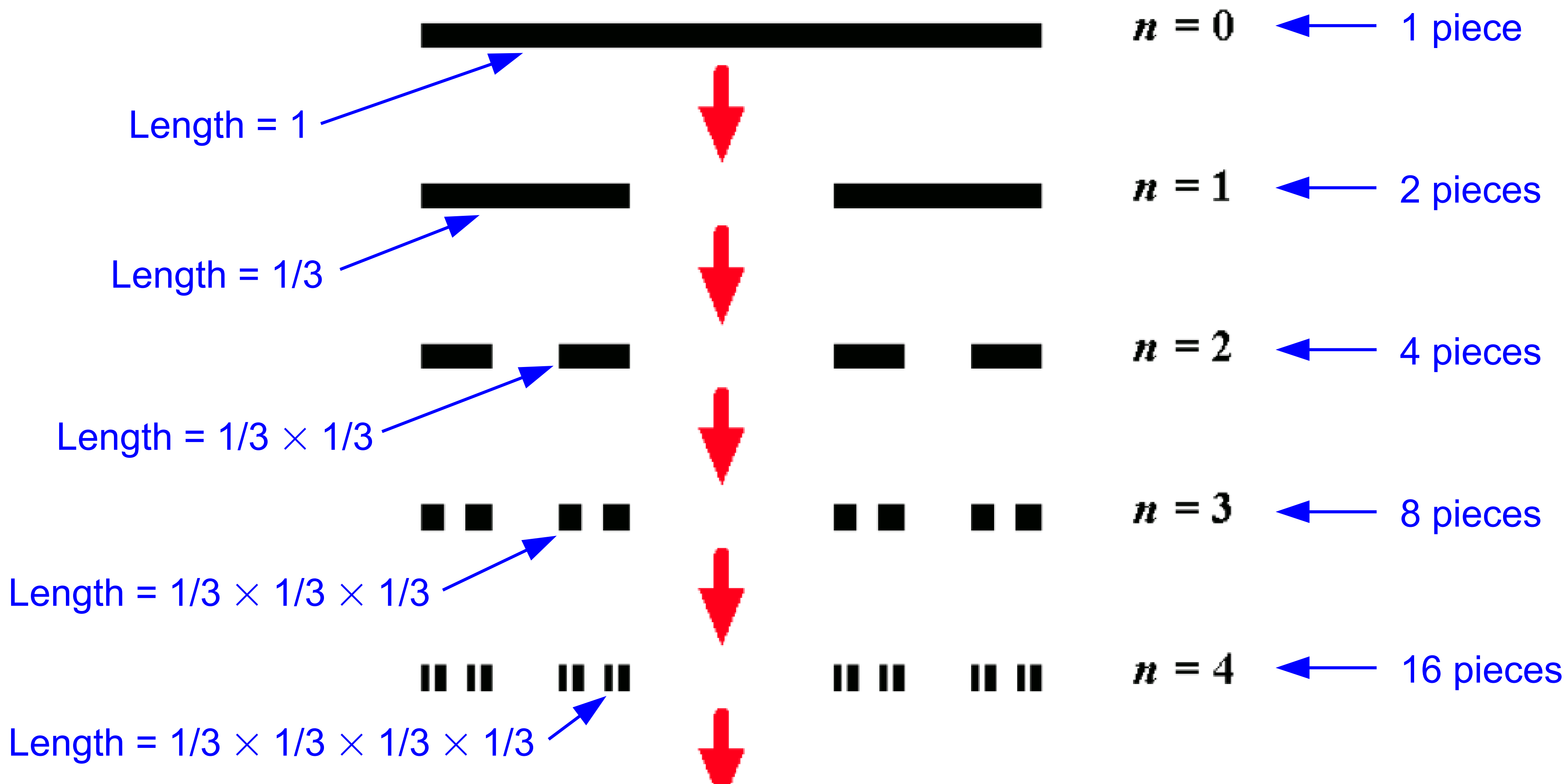
# The Cantor Set



# The Cantor Set

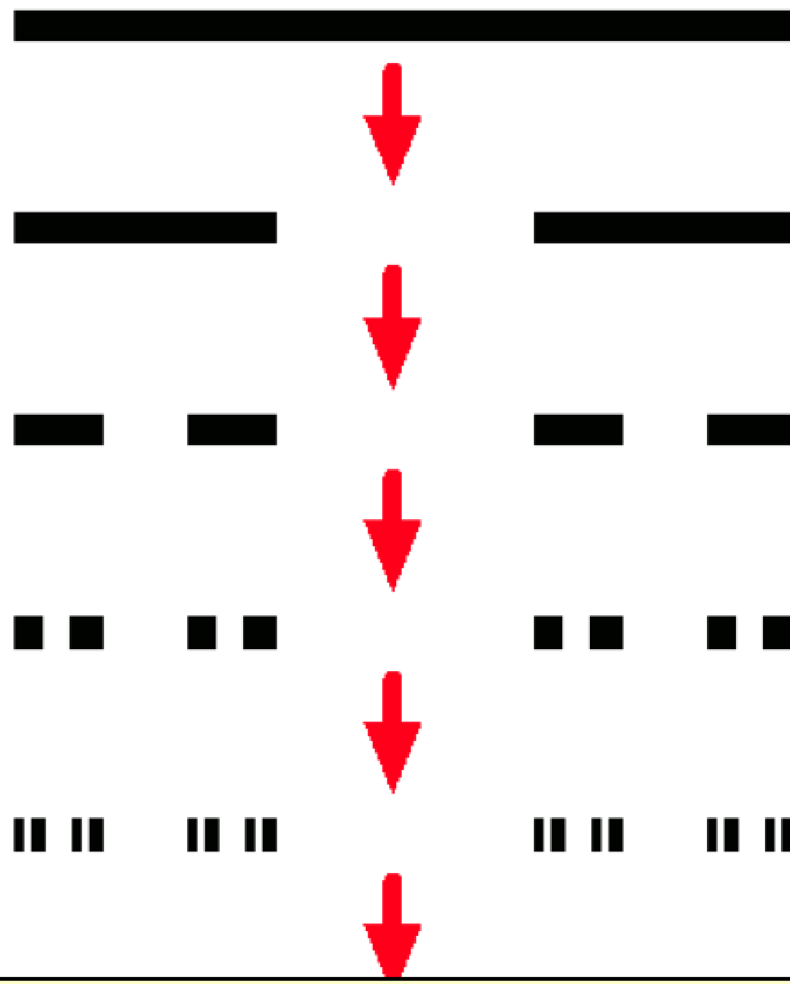


# The Cantor Set




$$\boxed{\text{Total length} = (1/3)^n \times 2^n = (2/3)^n}$$

# The Cantor Set




Total length $= (1/3)^n \times 2^n = (2/3)^n$

# The Cantor Set

As $n$ goes to infinity, $(2/3)^n$ goes to 0

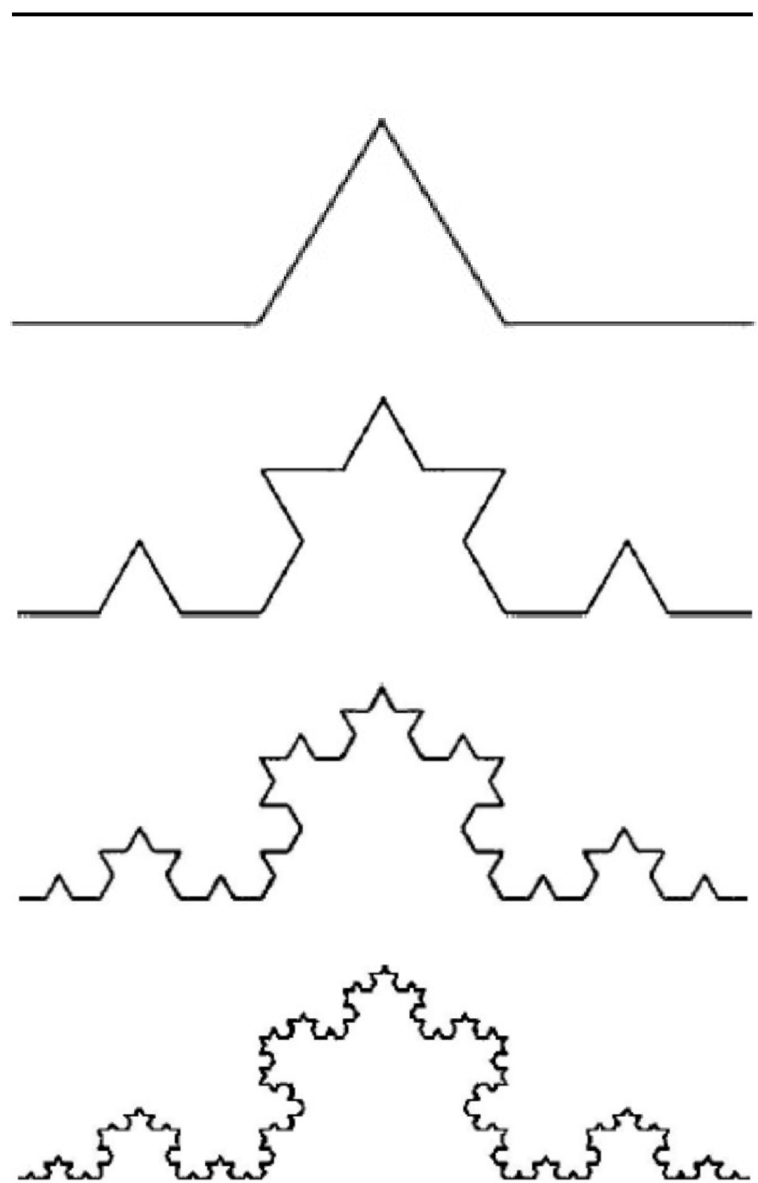
| $n$ | $(2/3)^n$ | $n$ | $(2/3)^n$ |
|---|---|---|---|
| 0 | 1.0000000 | 16 | 0.0015224 |
| 1 | 0.6666667 | 17 | 0.0010150 |
| 2 | 0.4444444 | 18 | 0.0006766 |
| 3 | 0.2962963 | 19 | 0.0004511 |
| 4 | 0.1975309 | 20 | 0.0003007 |
| 5 | 0.1316872 | 21 | 0.0002005 |
| 6 | 0.0877915 | 22 | 0.0001337 |
| 7 | 0.0585277 | 23 | 0.0000891 |
| 8 | 0.0390184 | 24 | 0.0000594 |
| 9 | 0.0260123 | 25 | 0.0000396 |
| 10 | 0.0173415 | 26 | 0.0000264 |
| 11 | 0.0115610 | 27 | 0.0000176 |
| 12 | 0.0077073 | 28 | 0.0000117 |
| 13 | 0.0051382 | 29 | 0.0000078 |
| 14 | 0.0034255 | 30 | 0.0000052 |
| 15 | 0.0022837 | | . . . *etc* . . . |

# The Cantor Set

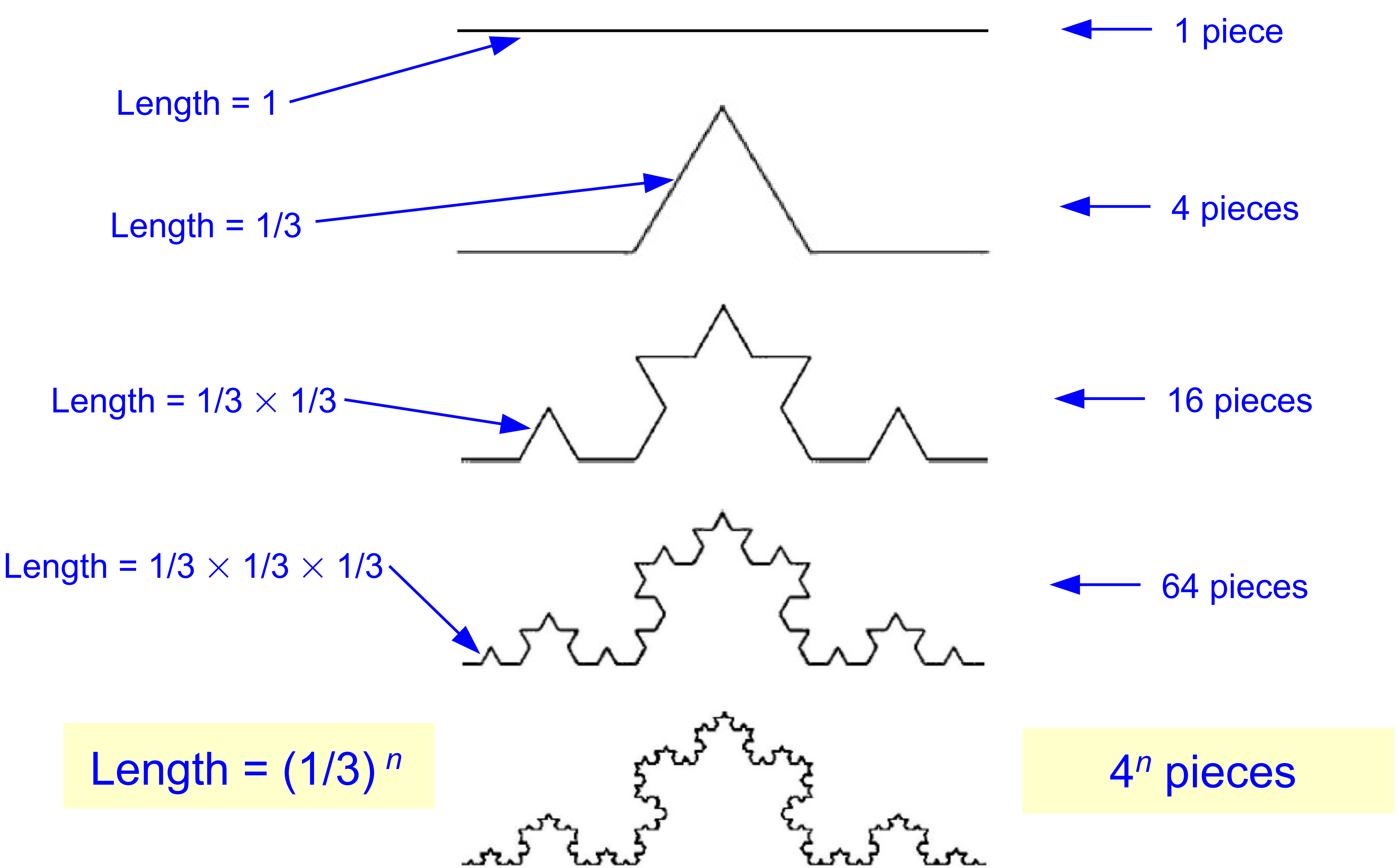
As $n$ goes to infinity, $(2/3)^n$ goes to 0

- The Cantor Set is defined as the **limit** of this process
- So the total “length” of the Cantor Set is **precisely zero**!
- Even though it contains an **infinite** number of points!

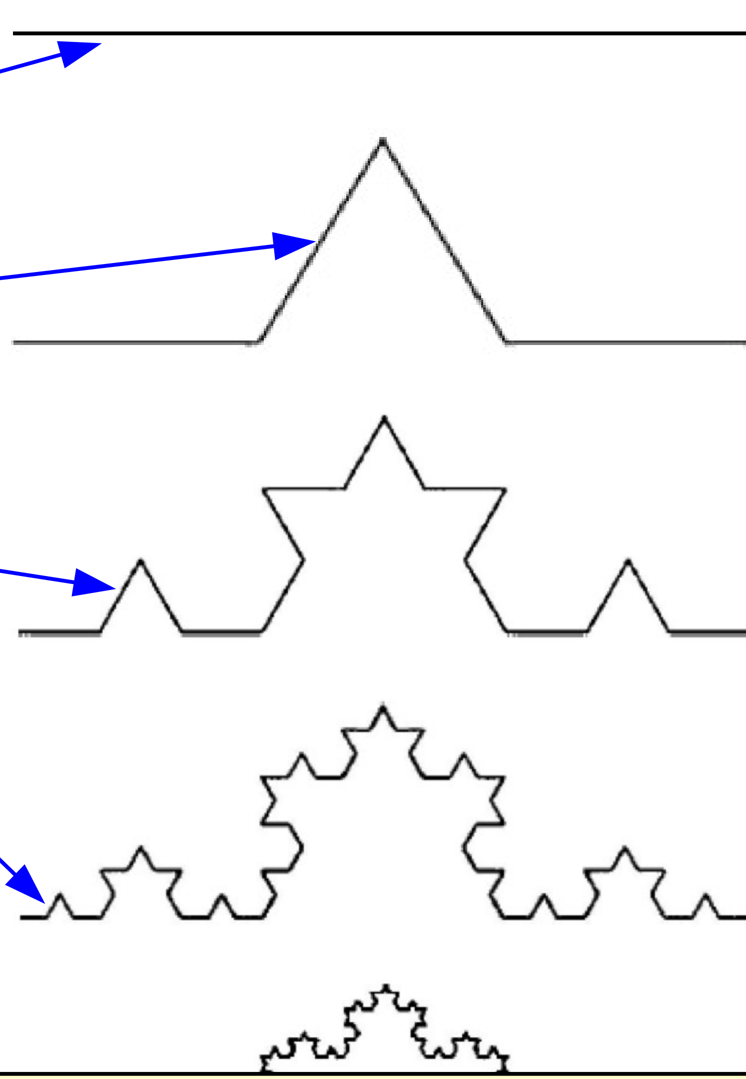
# The Koch Curve



# The Koch Curve



# The Koch Curve



Length = 1 | 1 piece

Length = 1/3 | 4 pieces

Length = $1/3 \times 1/3$ | 16 pieces

Length = $1/3 \times 1/3 \times 1/3$ | 64 pieces

Length = $(1/3)^n$ | $4^n$ pieces

Total length $= (1/3)^n \times 4^n = (4/3)^n$

# The Koch Curve

As $n$ goes to infinity, $(4/3)^n$ goes to infinity

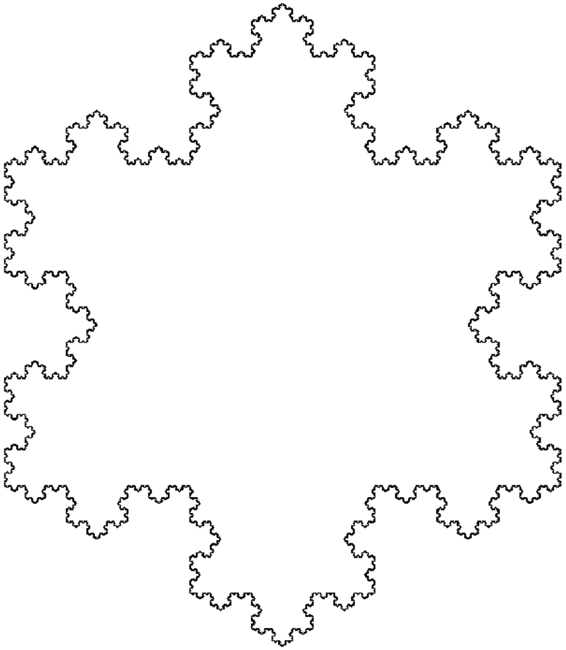
| $n$ | $(4/3)^n$ | $n$ | $(4/3)^n$ | $n$ | $(4/3)^n$ |
|---|---|---|---|---|---|
| 0 | 1.000 | 16 | 99.775 | 32 | 9954.961 |
| 1 | 1.333 | 17 | 133.033 | 33 | 13273.282 |
| 2 | 1.778 | 18 | 177.377 | 34 | 17697.709 |
| 3 | 2.370 | 19 | 236.503 | 35 | 23596.945 |
| 4 | 3.160 | 20 | 315.337 | 36 | 31462.593 |
| 5 | 4.214 | 21 | 420.449 | 37 | 41950.125 |
| 6 | 5.619 | 22 | 560.599 | 38 | 55933.499 |
| 7 | 7.492 | 23 | 747.465 | 39 | 74577.999 |
| 8 | 9.989 | 24 | 996.620 | 40 | 99437.332 |
| 9 | 13.318 | 25 | 1328.827 | 41 | 132583.110 |
| 10 | 17.758 | 26 | 1771.769 | 42 | 176777.480 |
| 11 | 23.677 | 27 | 2362.359 | 43 | 235703.306 |
| 12 | 31.569 | 28 | 3149.812 | 44 | 314271.075 |
| 13 | 42.092 | 29 | 4199.749 | 45 | 419028.100 |
| 14 | 56.123 | 30 | 5599.666 | 46 | 558704.133 |
| 15 | 74.831 | 31 | 7466.221 | 47 | 744938.844 |

. . . *etc* . . .

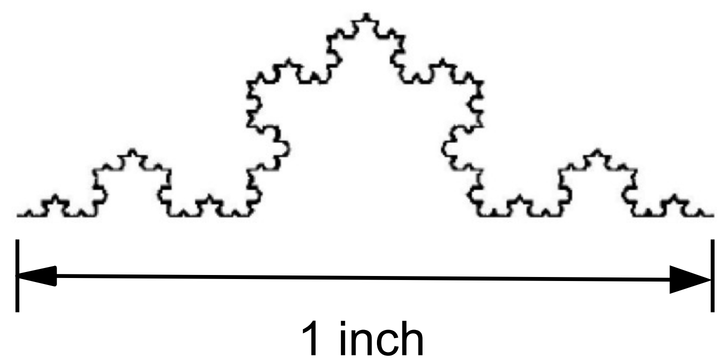
# The Koch Curve

As $n$ goes to infinity, $(4/3)^n$ goes to infinity

- The Koch Curve is defined as the **limit** of this process
- So the total length of the Koch Curve is **infinite**
- The perimeter of a Koch Snowflake is also **infinite**, but it encloses a **finite area**



# The Koch Curve




- A 1-inch wide Koch curve iterated for 100 steps would be over **49 million miles long** if fully stretched out!

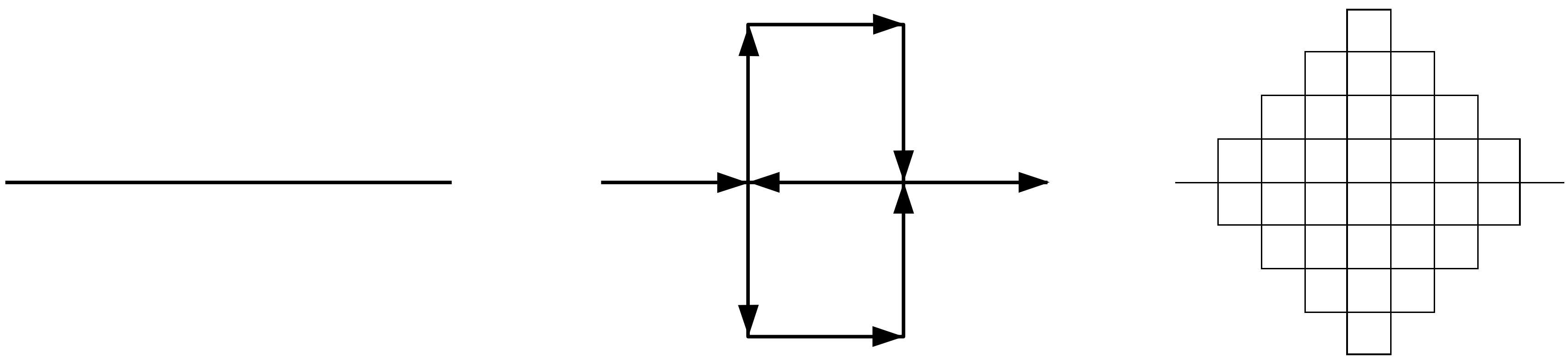
$(4/3)^{100}$ = 3,117,982,410,207.9 inches

= 259,831,867,517.3 feet

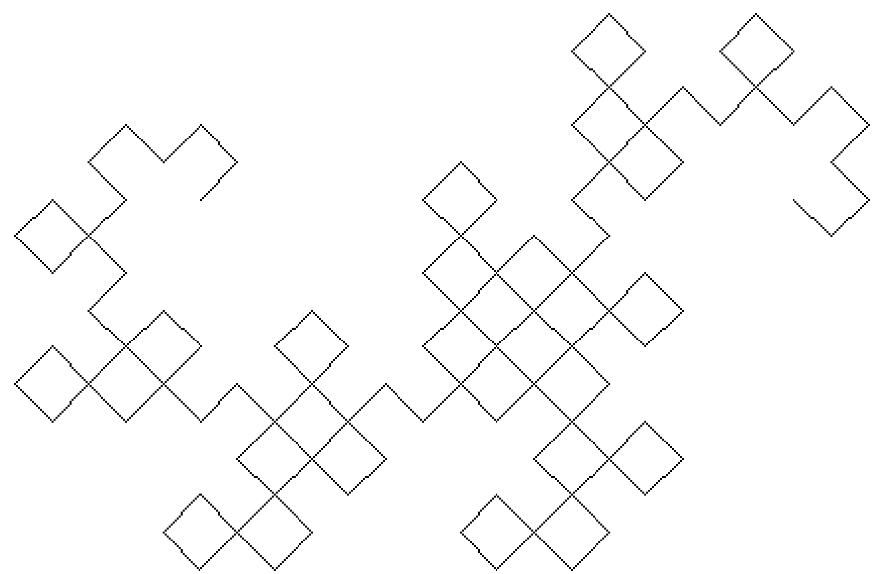
= 49,210,580.9 miles

# Space-Filling Curves

- Peano Curve



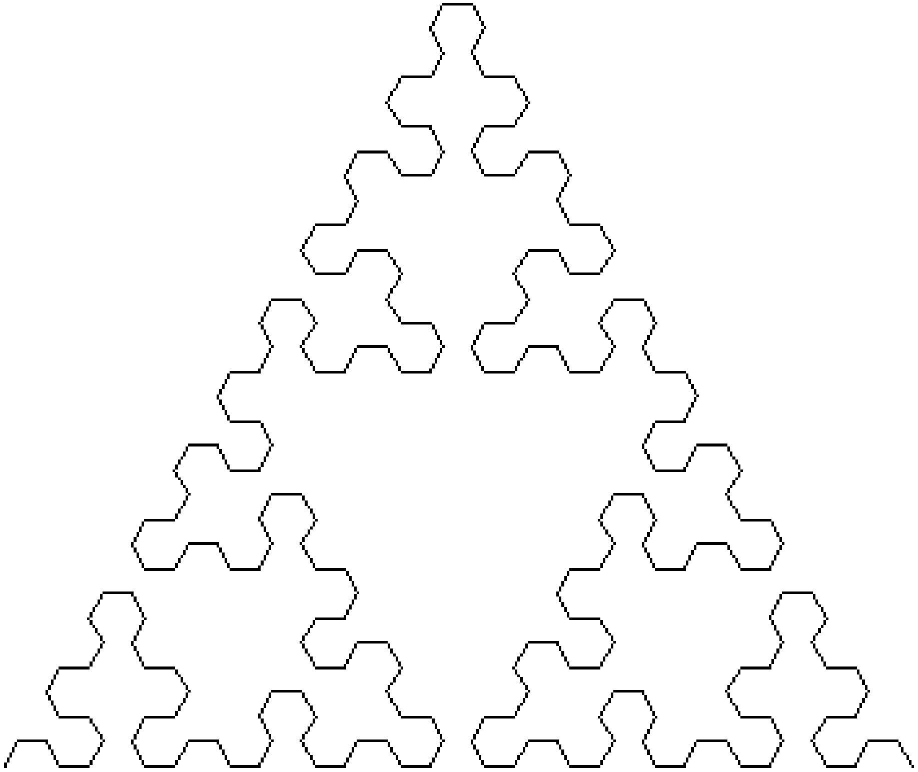
- Dragon Curve



# Space-Filling Curves

- Sierpinski Arrowhead

- Can be drawn as a single continuous line that never crosses itself



# Computer Demos

- Peano Curve
- Dragon Curve
- Sierpinski Arrowhead

Are these curves **1-dimensional** or **2-dimensional**?