The Bifurcation Diagram for the Logistic Map

$$x_{t+1} = R\, x_t\, (1 - x_t)$$

What Happens Near R = 3 ?

- R = 2.9

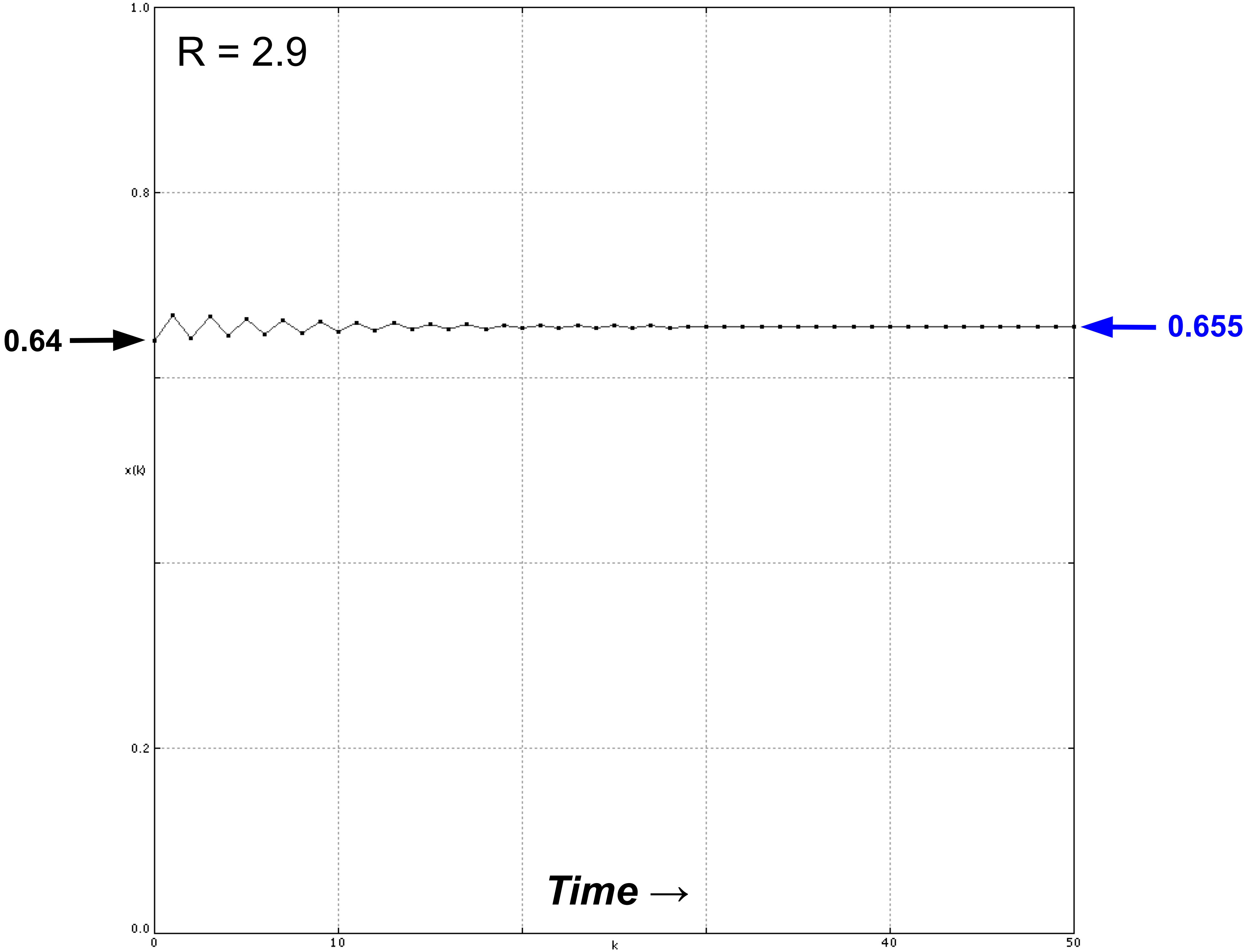
  x_0 = **0.64** → 0.6681600 → 0.6429944 → 0.6657025 → 0.6453738
  . . . → **0.6551724** → **0.6551724** → **0.6551724** → . . .

  Fixed point attractor ≈ 0.655

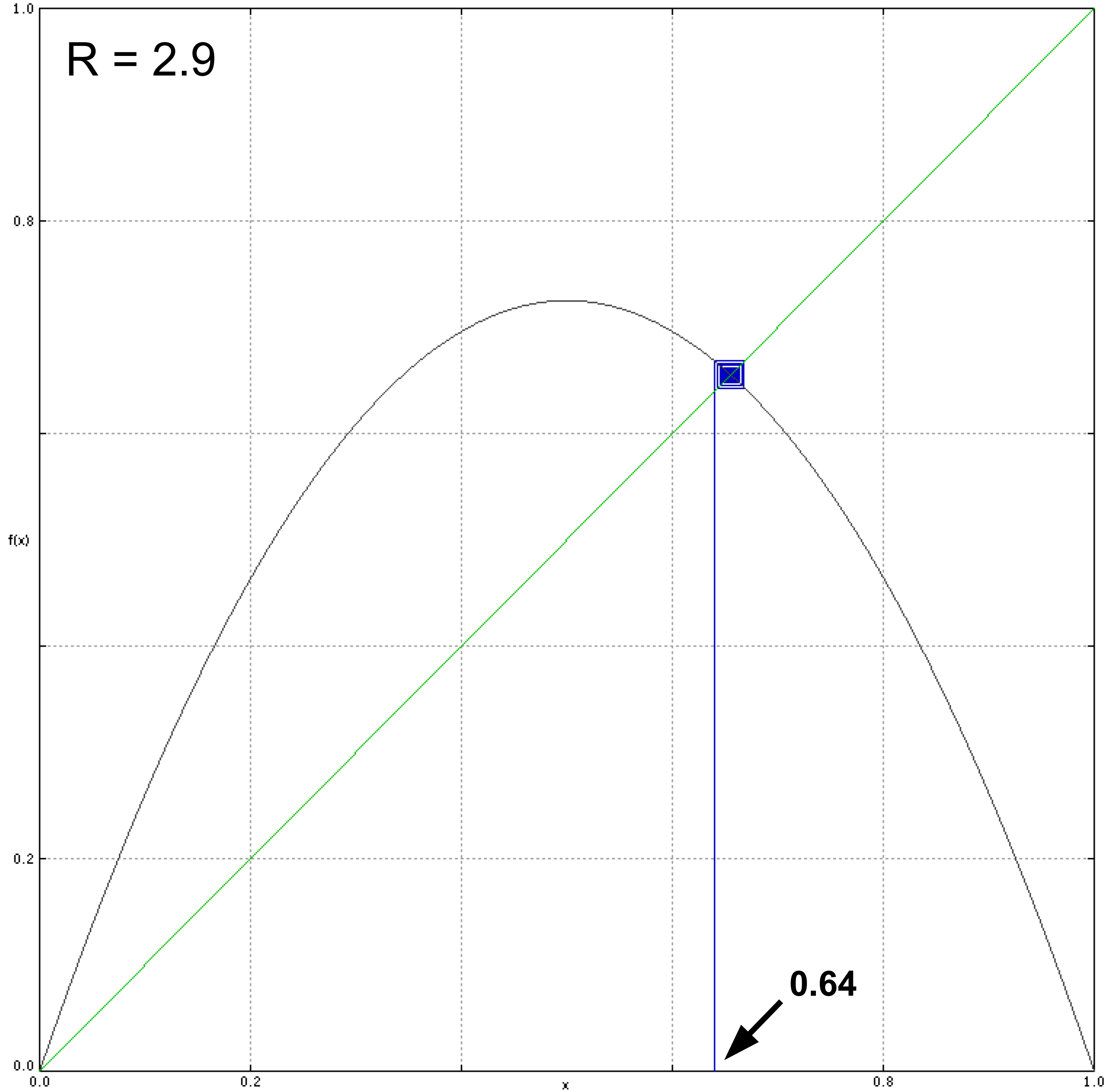
- R = 3.1

  x_0 = **0.64** → 0.7142400 → 0.6327138 → 0.7203999 → 0.6244141
  . . . → **0.5580141** → **0.7645665** → **0.5580141** → **0.7645665** . . .

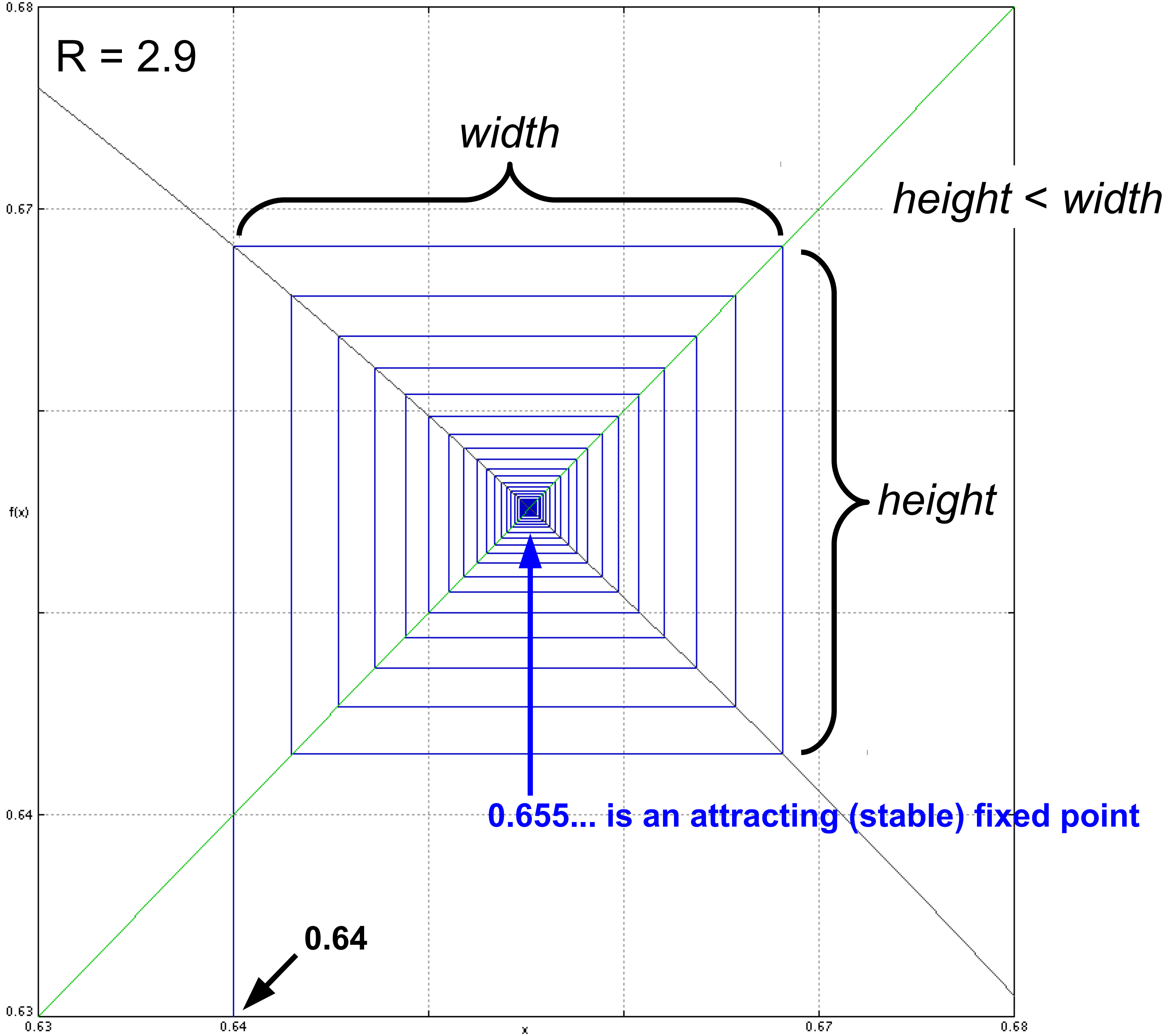
  Period-2 attractor ≈ 0.558 and 0.765

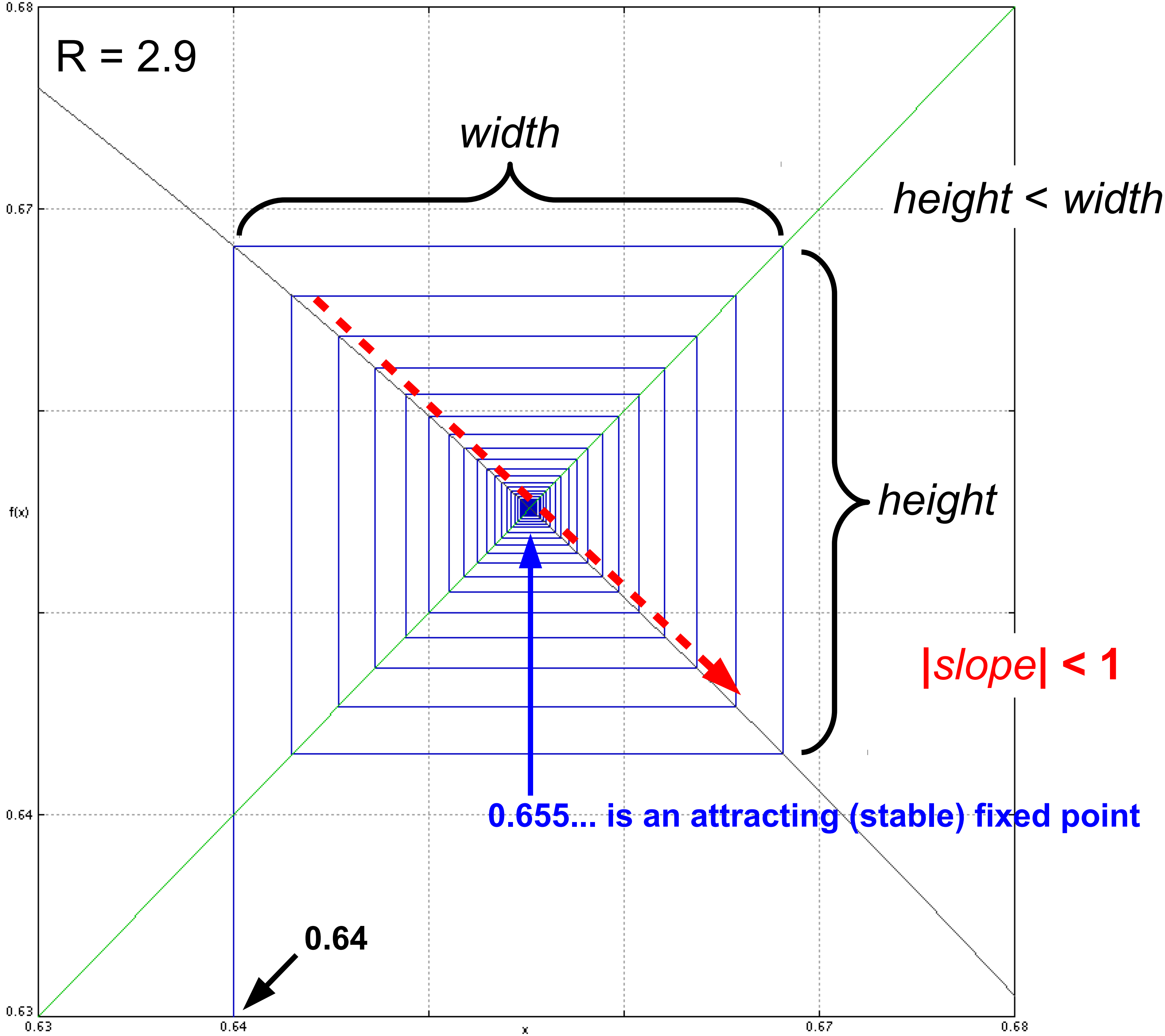


R = 2.9
0.64
0.655
Time →
x(k)
k
1.0
0.8
0.2
0.0
0
10
40
50



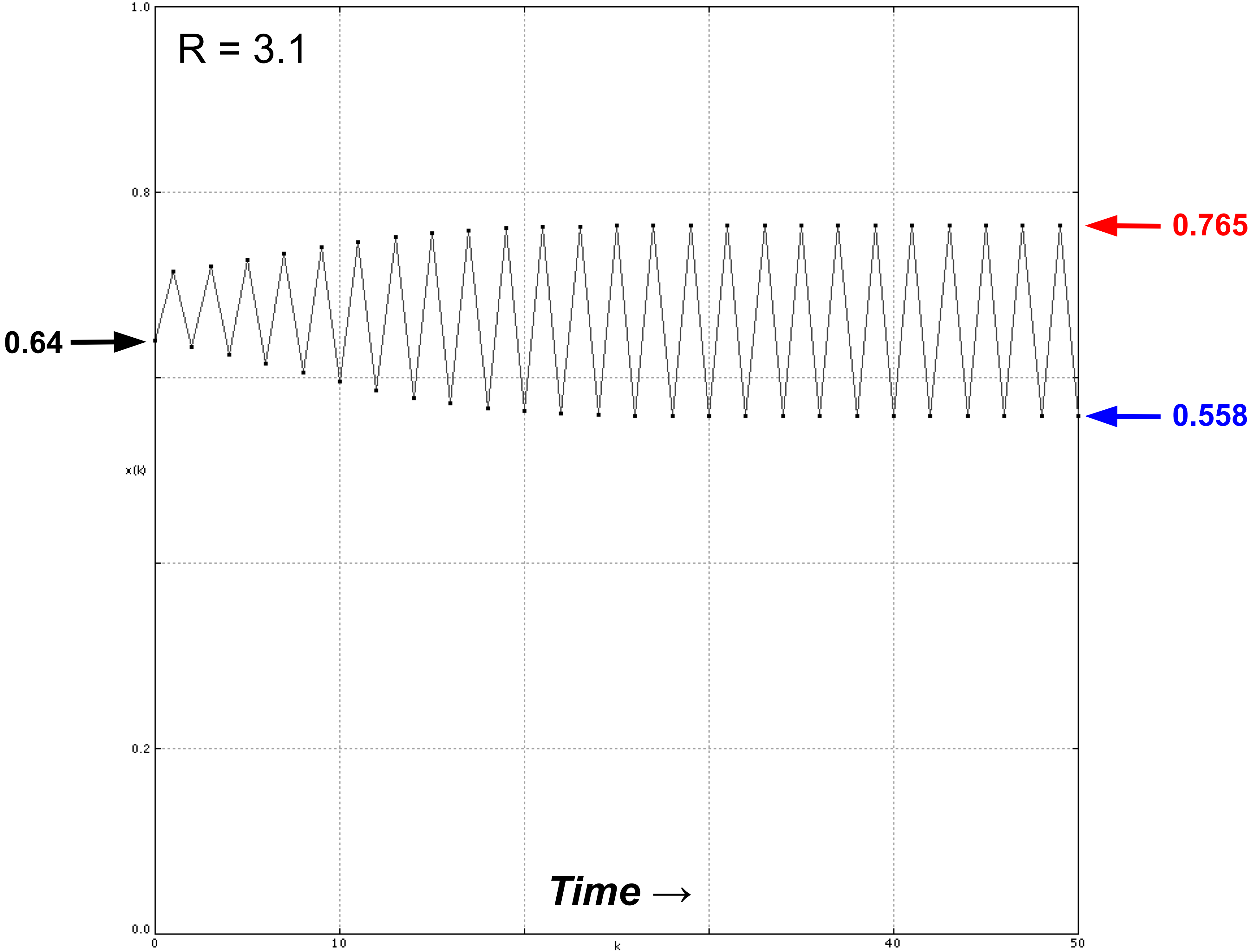
R = 2.9
1.0
0.8
f(x)
0.2
0.0
0.0
0.2
x
0.8
1.0
0.64


R = 2.9
width
height < width
height
0.655... is an attracting (stable) fixed point
0.64
f(x)
x
0.63
0.64
0.67
0.68

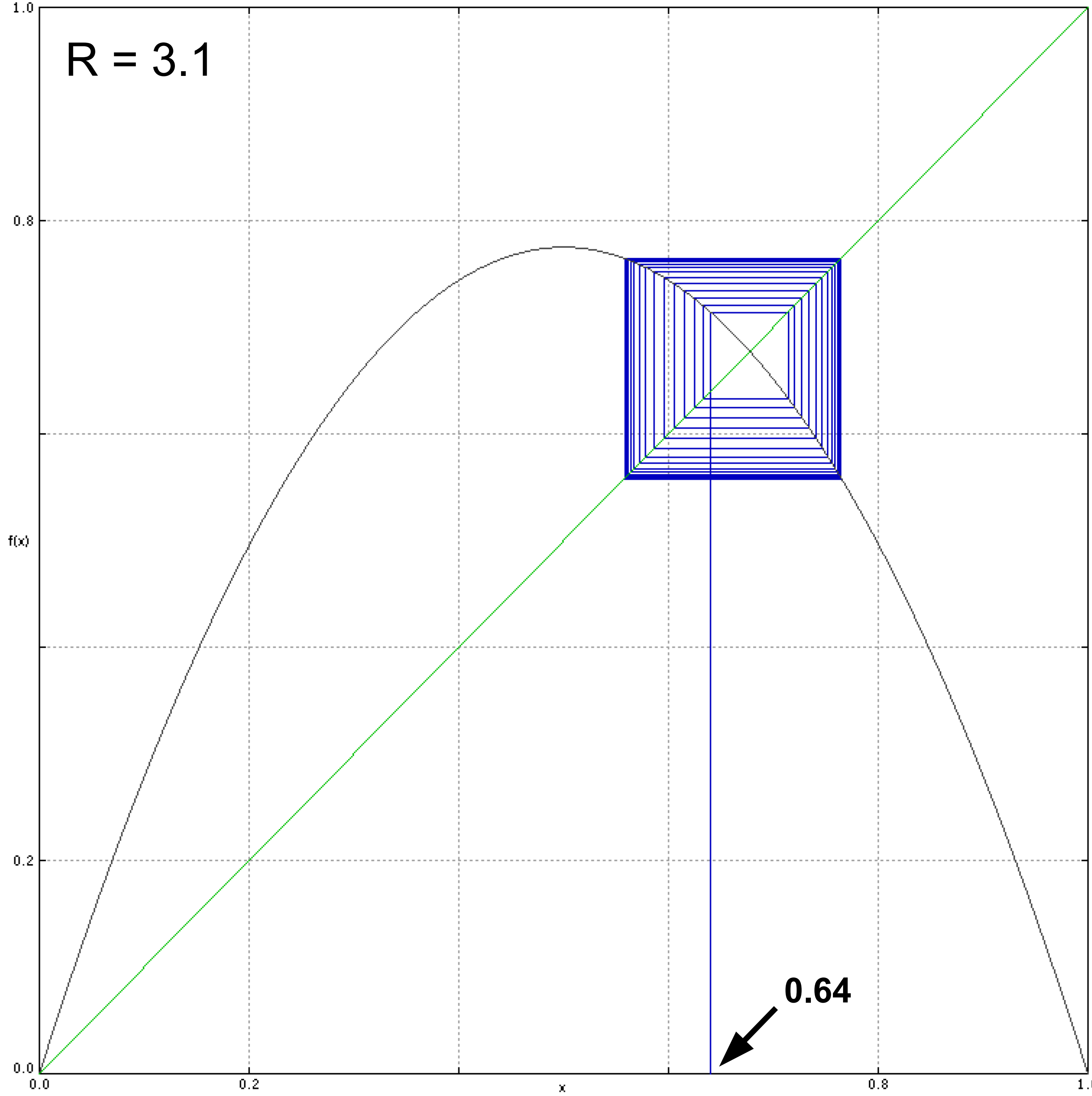


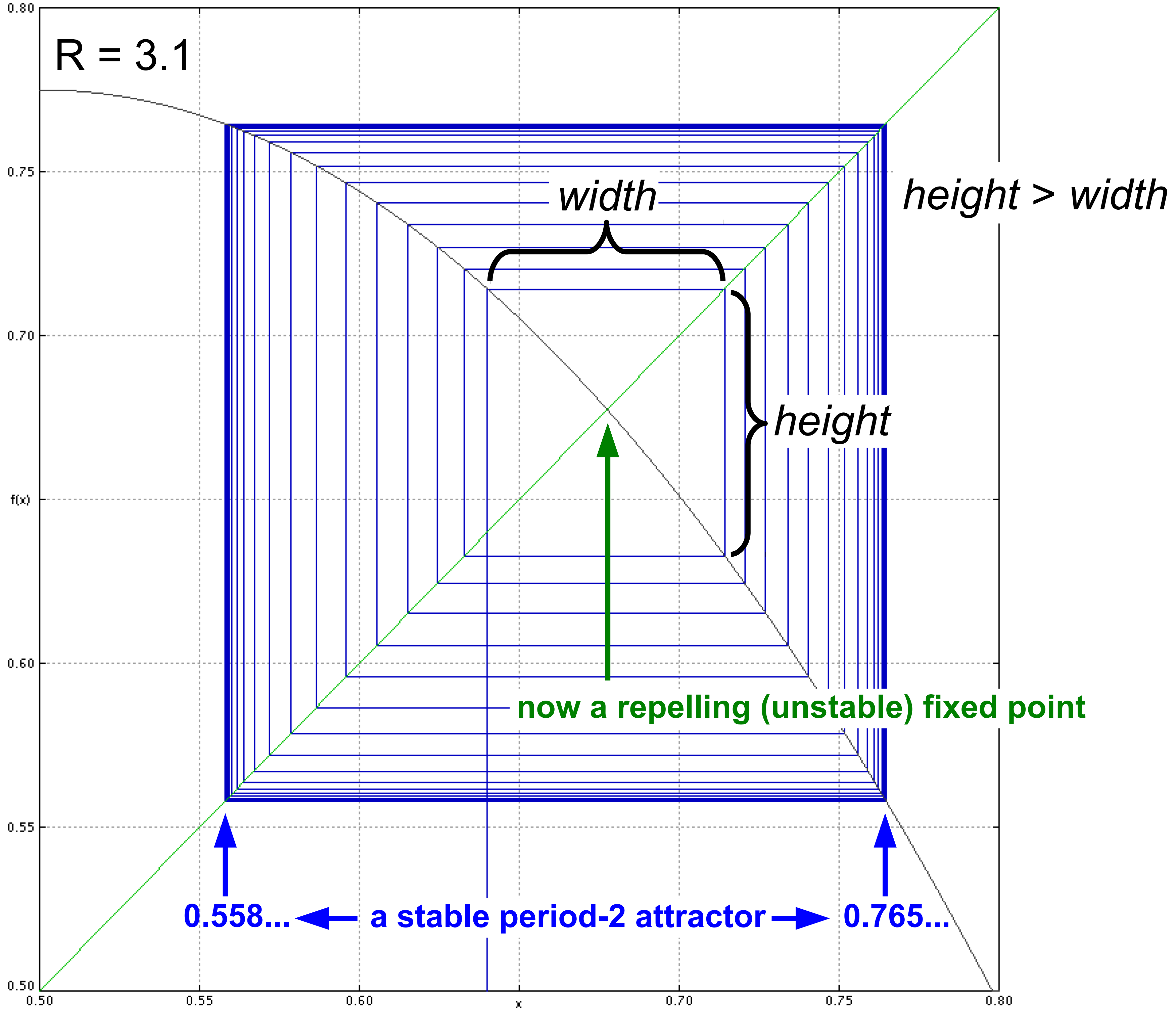
R = 2.9
width
height < width
height
|slope| < 1
0.655... is an attracting (stable) fixed point
0.64
f(x)
x
0.63
0.64
0.67
0.68

Now we will turn R up to 3.1

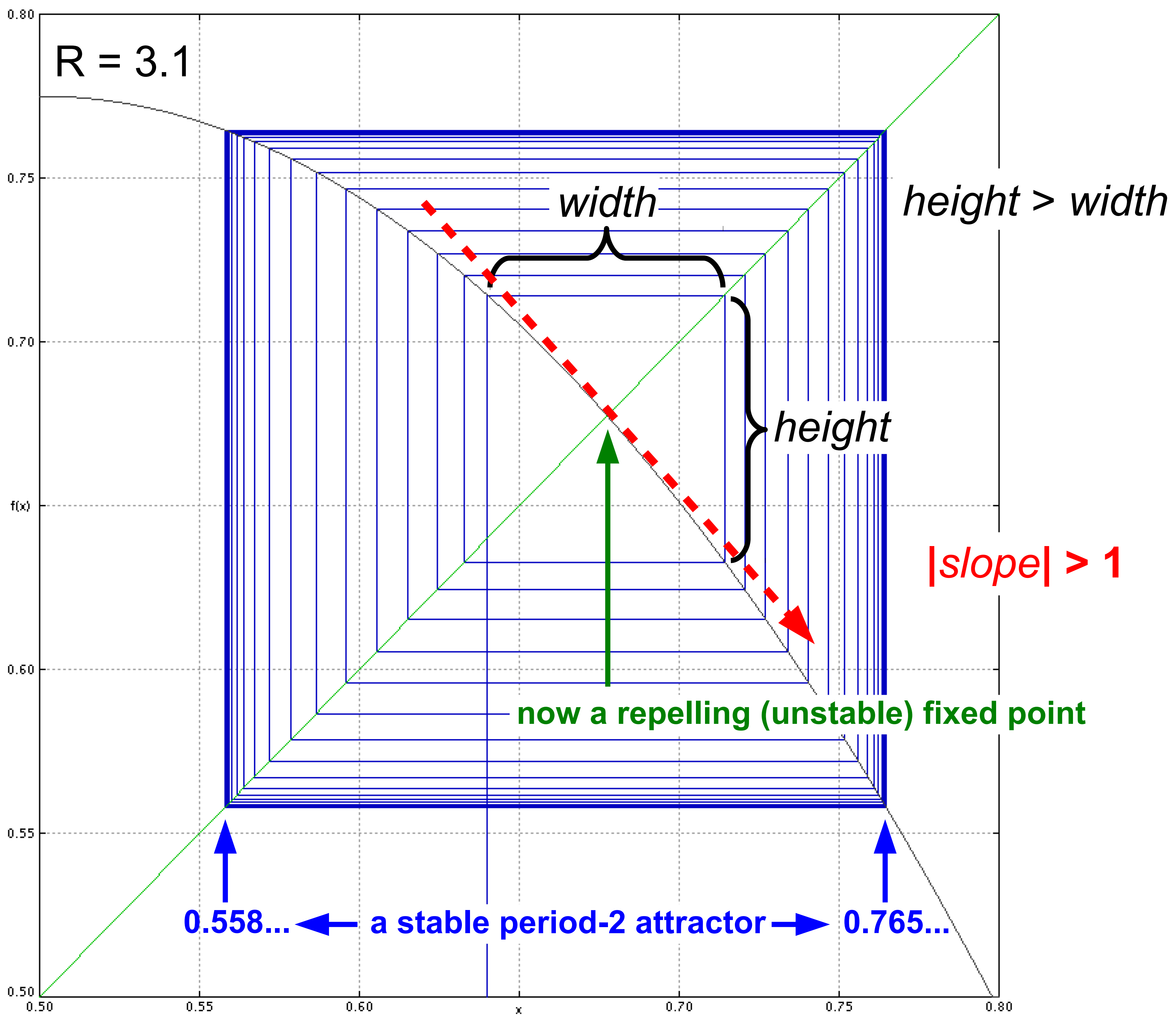


R = 3.1
0.64
0.765
0.558
1.0
0.8
0.2
0.0
x(k)
0
10
40
50
k
Time →


R = 3.1
1.0
0.8
f(x)
0.2
0.0
0.0
0.2
x
0.8
1.0
0.64



R = 3.1
width
height > width
height
now a repelling (unstable) fixed point
0.558... a stable period-2 attractor 0.765...
f(x)
x
0.50
0.55
0.60
0.70
0.75
0.80



R = 3.1
width
height > width
height
|slope| > 1
now a repelling (unstable) fixed point
0.558...
a stable period-2 attractor
0.765...
f(x)
x
0.50
0.55
0.60
0.70
0.75
0.80

What Just Happened?

- A stable fixed point attractor (period-1) **bifurcated** into a stable period-2 attractor as R went from 2.9 to 3.1

- The **period** of the attractor **doubled**

- The stable attracting fixed point became an **unstable repelling** fixed point

- The bifurcation happens at exactly R = 3.0, when the **slope** of the tangent line changes from < 1 to > 1

Period-2 Attractor for R = 3.2

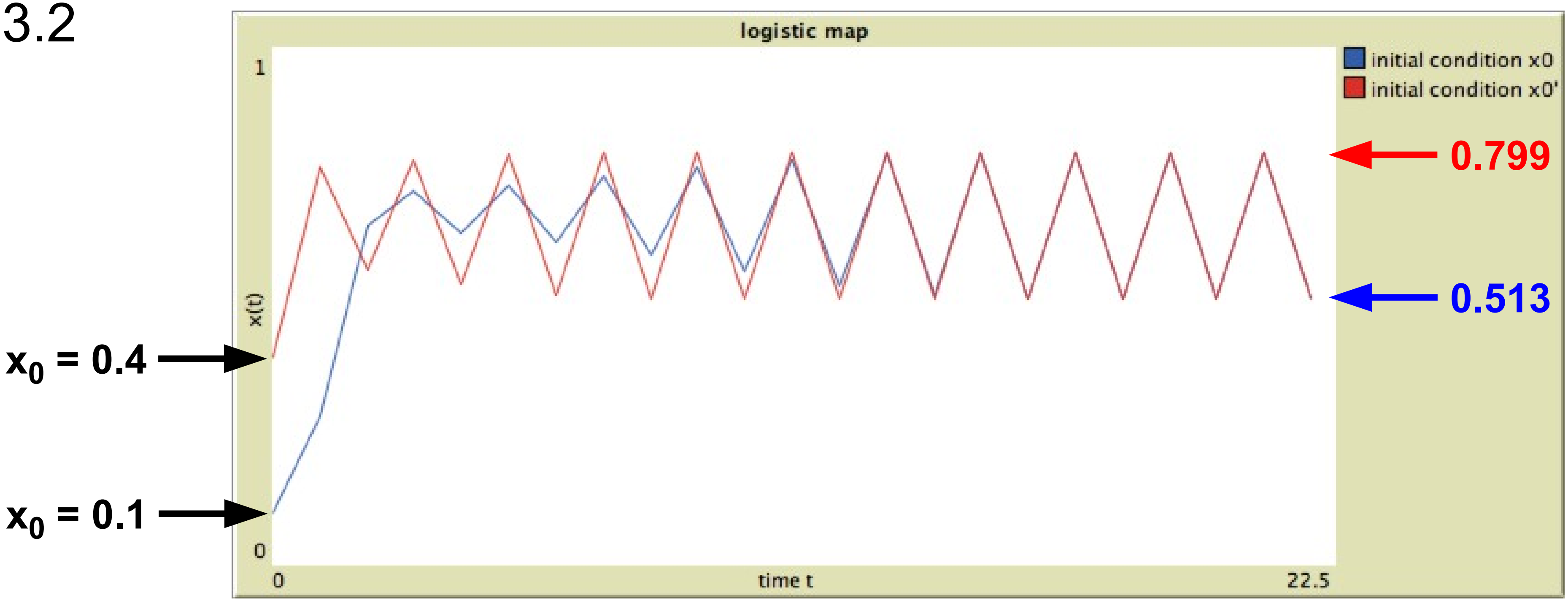
x_0 = **0.1** → 0.288 → 0.656 → 0.722 → 0.642 → 0.735 →
. . . → **0.513** → **0.799** → **0.513** → **0.799** → **0.513** . . .

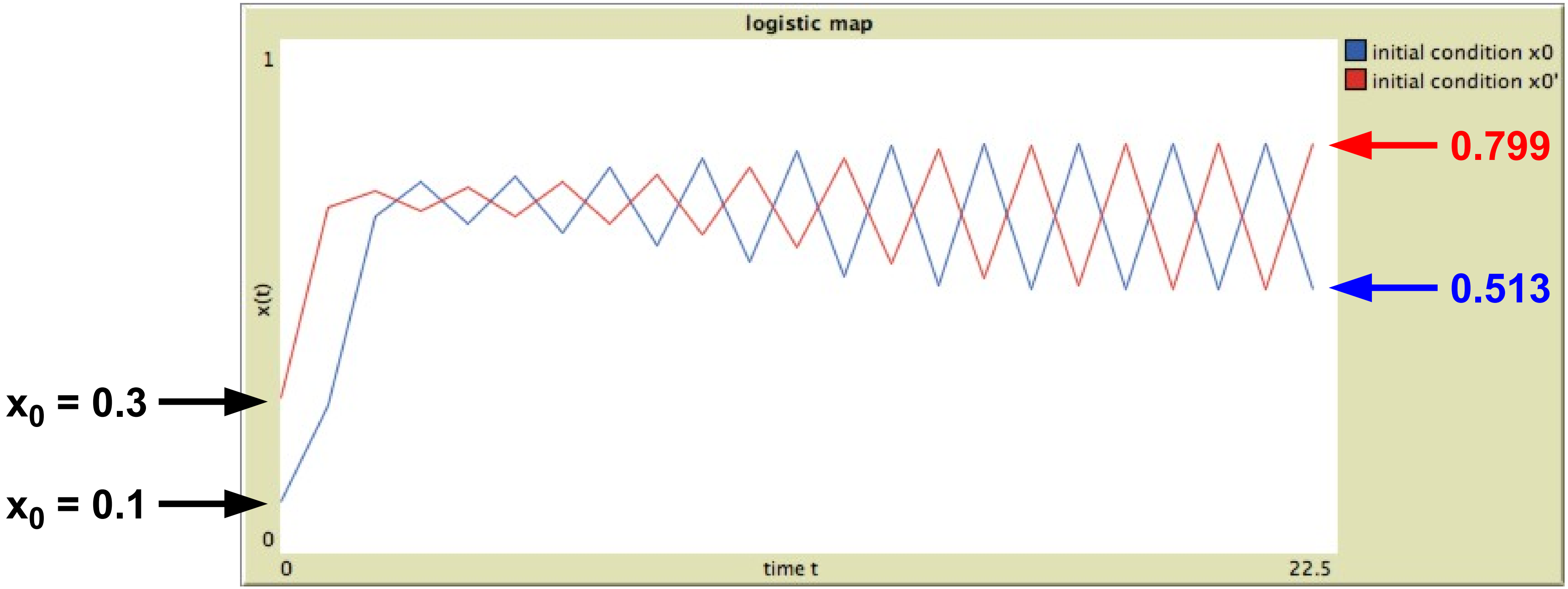
x_0 = **0.2** → 0.512 → 0.800 → 0.513 → 0.799 → 0.513 →
. . . → **0.799** → **0.513** → **0.799** → **0.513** → **0.799** . . .

x_0 = **0.3** → 0.672 → 0.705 → 0.665 → 0.713 → 0.656 →
. . . → **0.799** → **0.513** → **0.799** → **0.513** → **0.799** . . .

x_0 = **0.4** → 0.768 → 0.570 → 0.784 → 0.541 → 0.795 →
. . . → **0.513** → **0.799** → **0.513** → **0.799** → **0.513** . . .

R = 3.2


logistic map
1
0
x(t)
0
time t
22.5
initial condition x0
initial condition x0'
0.799
0.513
x_0 = 0.4
x_0 = 0.1


logistic map
1
0
x(t)
0
time t
22.5
initial condition x0
initial condition x0'
0.799
0.513
x_0 = 0.3
x_0 = 0.1

Every Time Step: $x_t \rightarrow x_{t+1}$

x_0 = **0.1** → . . . → **0.513** → **0.799** → **0.513** → **0.799** → **0.513** . . .

x_0 = **0.2** → . . . → **0.799** → **0.513** → **0.799** → **0.513** → **0.799** . . .

x_0 = **0.3** → . . . → **0.799** → **0.513** → **0.799** → **0.513** → **0.799** . . .

x_0 = **0.4** → . . . → **0.513** → **0.799** → **0.513** → **0.799** → **0.513** . . .

Period-2 behavior

Every Other Time Step: $x_t \rightarrow x_{t+2}$

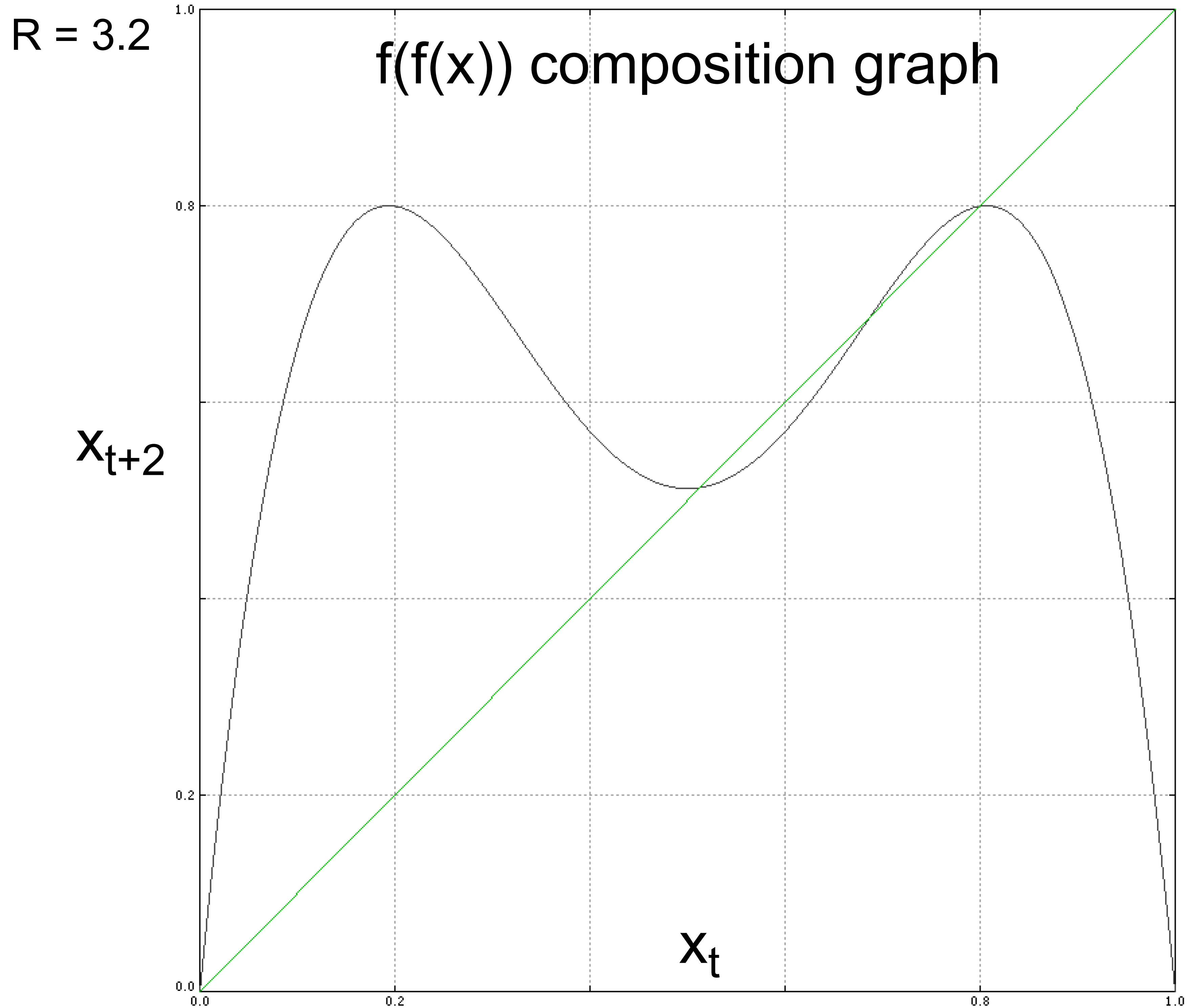
x_0 = **0.1** → . . . → **0.513** → **0.513** → **0.513** . . .

x_0 = **0.2** → . . . → **0.799** → **0.799** → **0.799** . . .

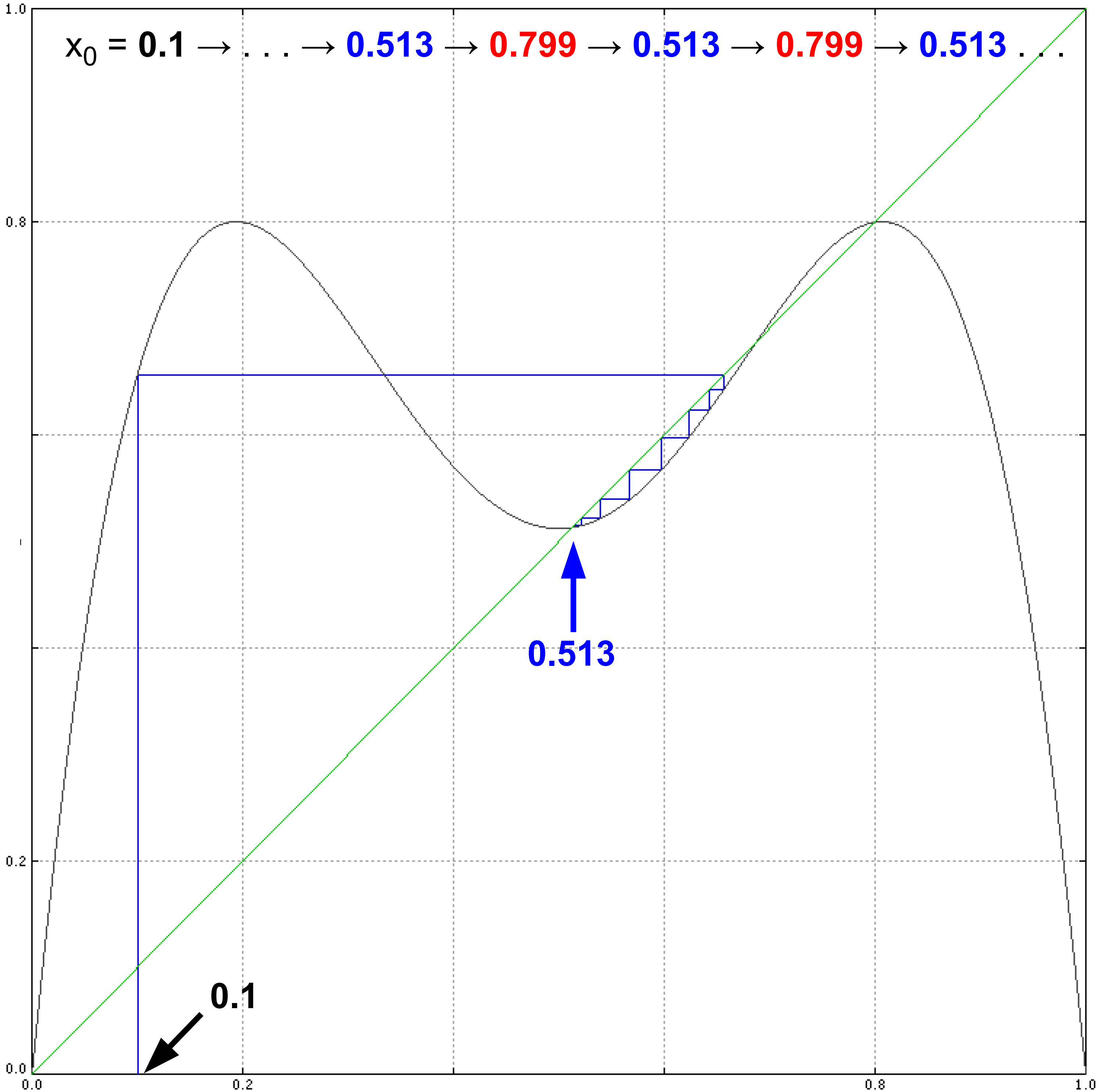
x_0 = **0.3** → . . . → **0.799** → **0.799** → **0.799** . . .

x_0 = **0.4** → . . . → **0.513** → **0.513** → **0.513** . . .

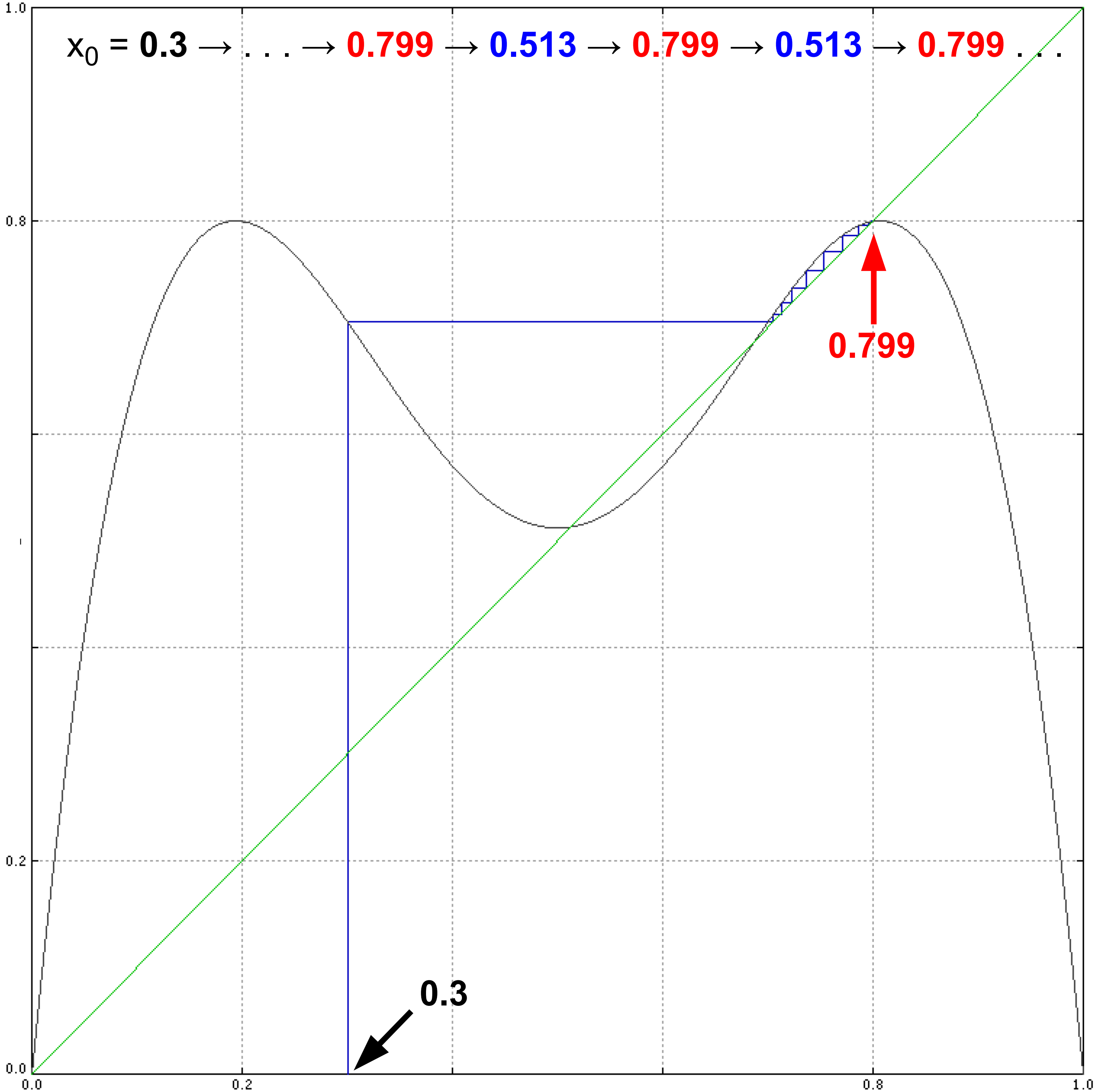
Fixed-point behavior


R = 3.2
f(f(x)) composition graph
x_{t+2}
x_t
1.0
0.8
0.2
0.0
0.0
0.2
0.8
1.0

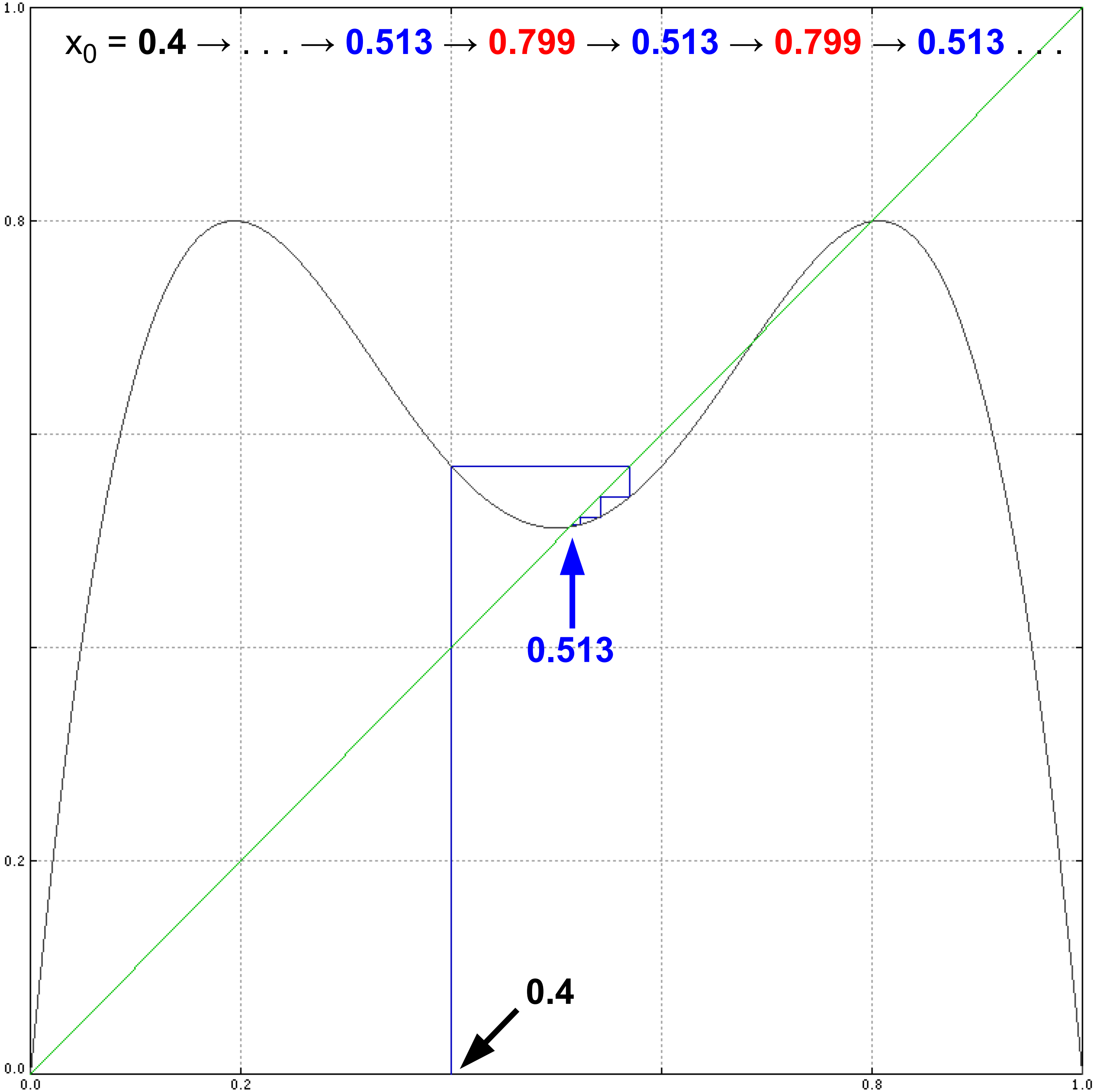
R = 3.2

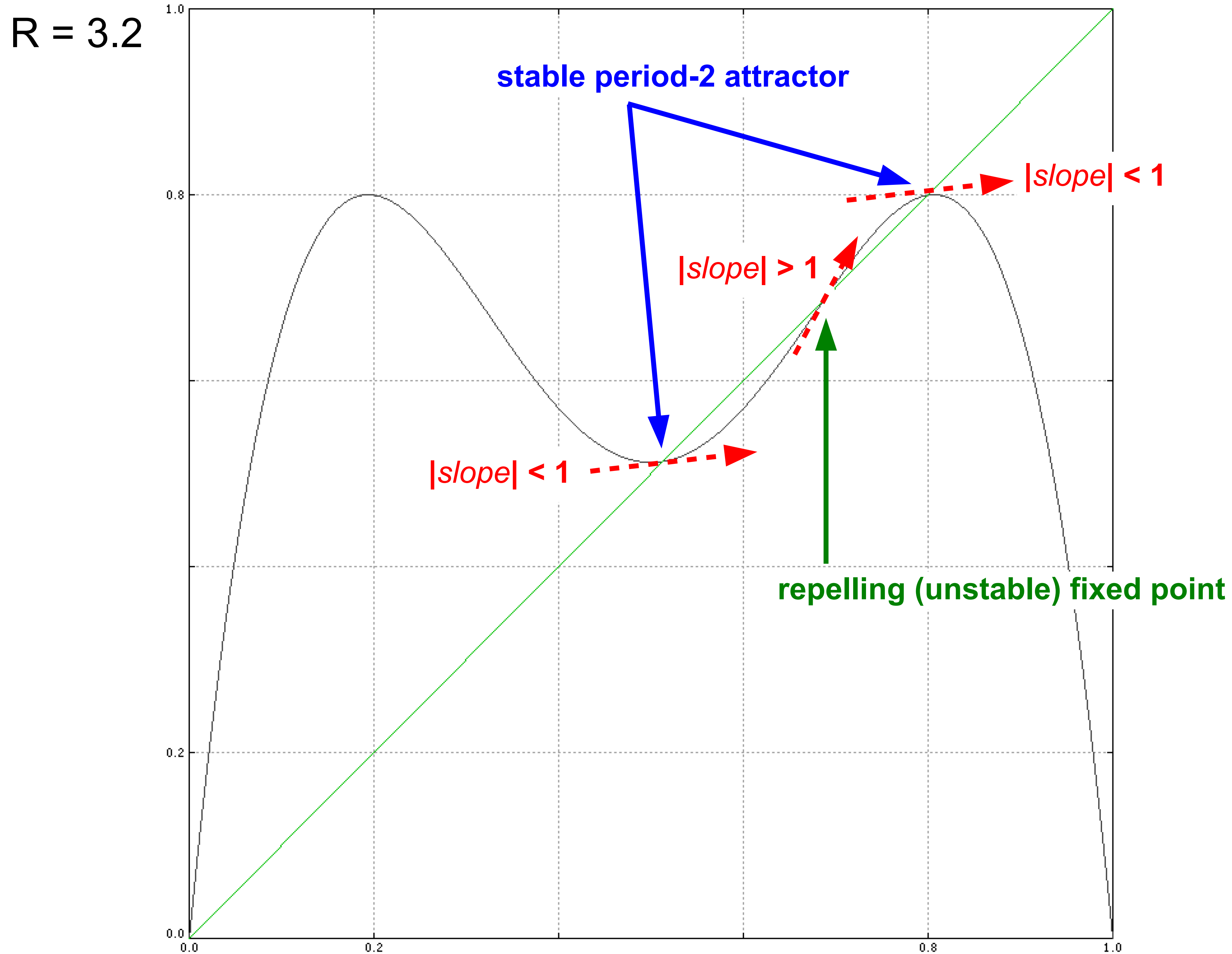


R = 3.2

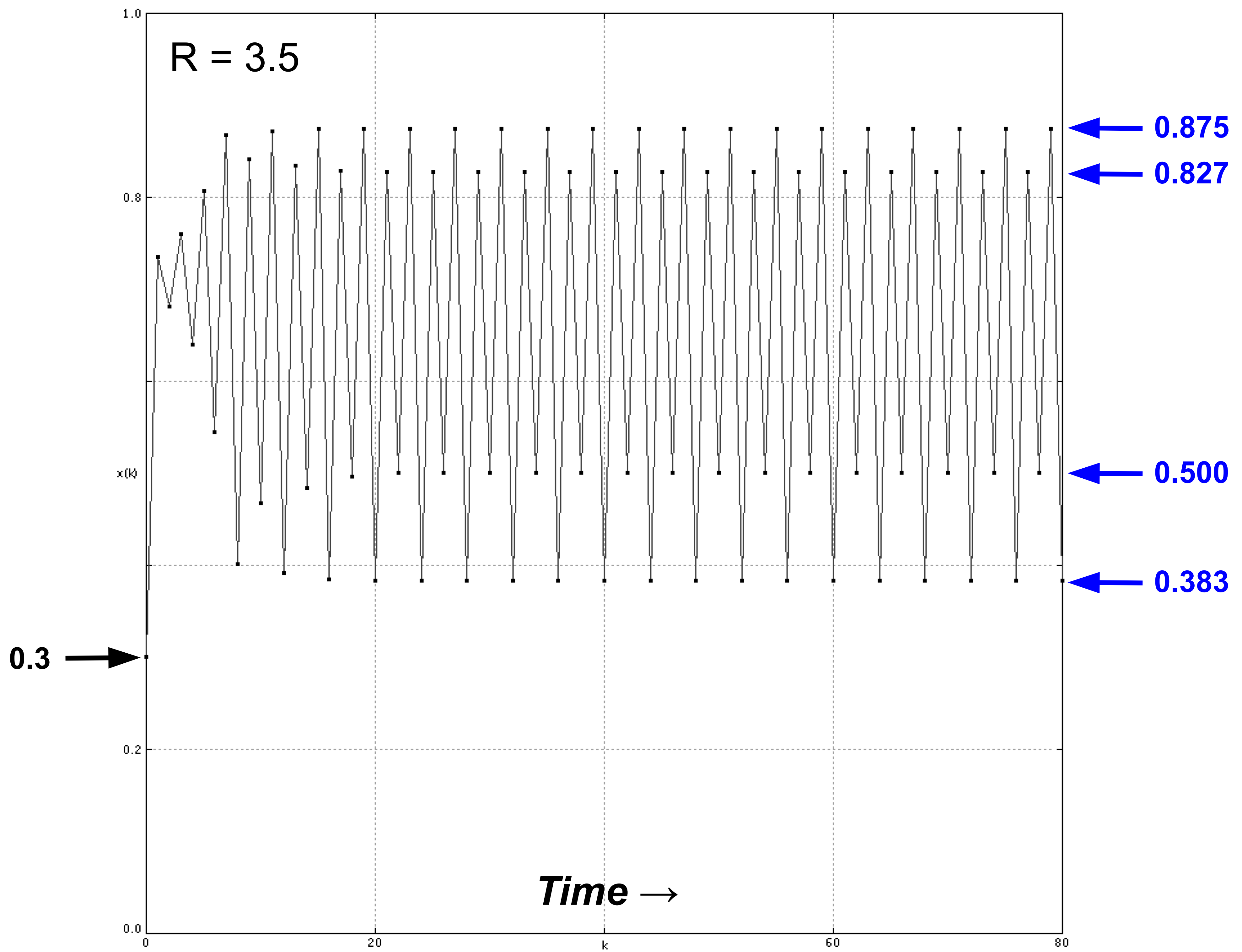


R = 3.2

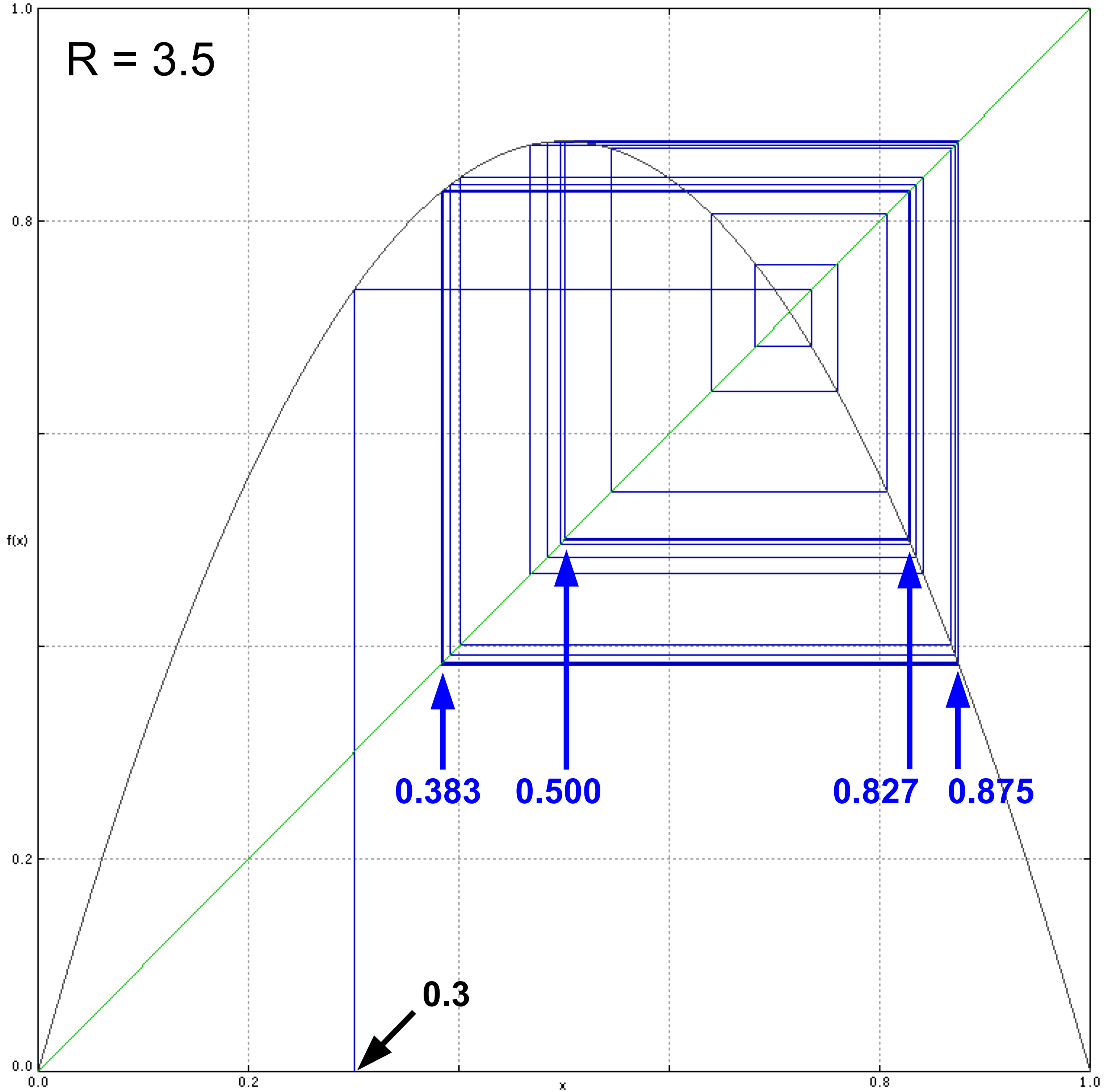


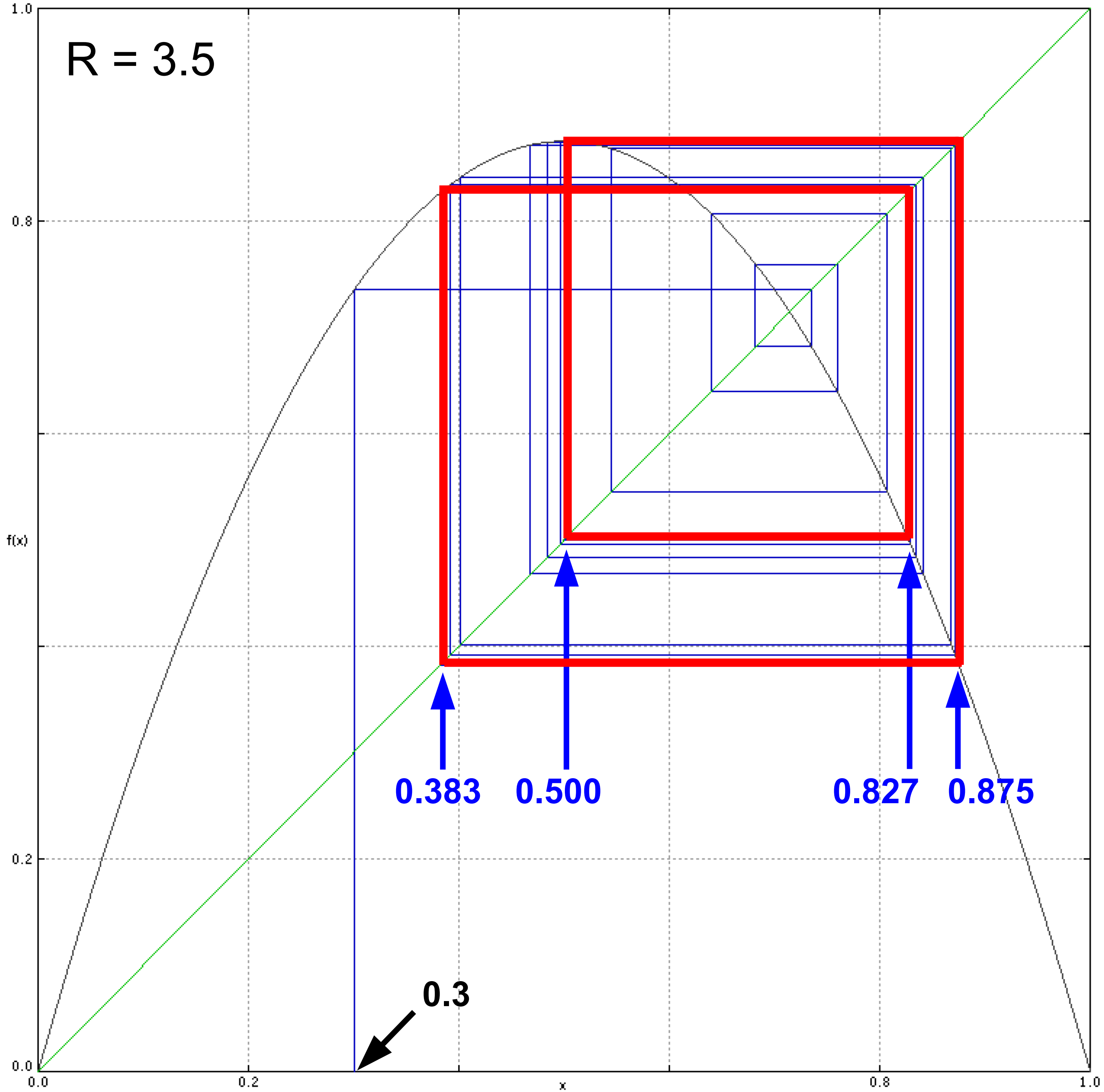

R = 3.2
stable period-2 attractor
|slope| < 1
|slope| > 1
|slope| < 1
repelling (unstable) fixed point
1.0
0.8
0.2
0.0
0.0
0.2
0.8
1.0

Now we will turn R up to 3.5



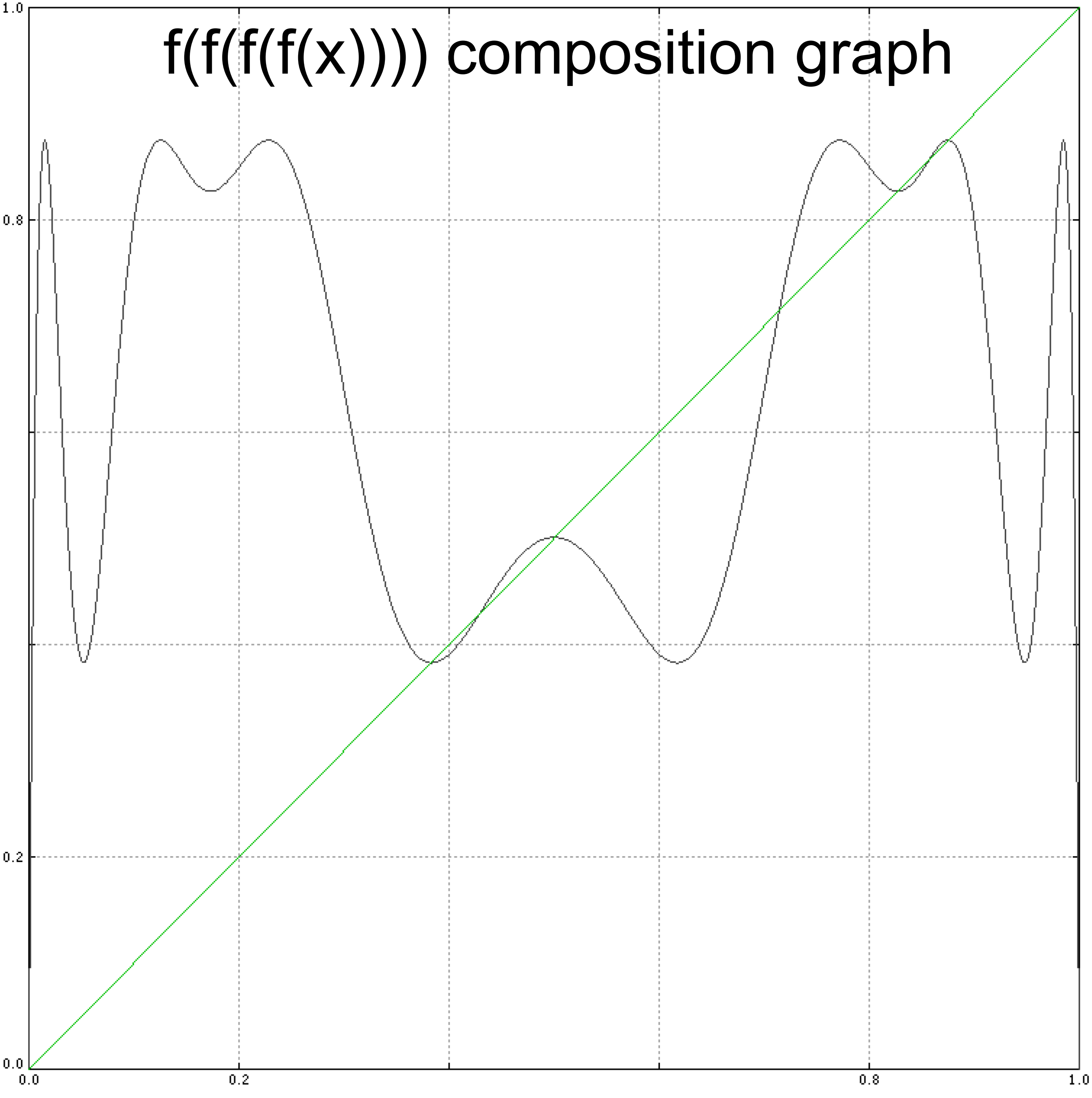
R = 3.5
0.875
0.827
0.500
0.383
0.3
Time →
1.0
0.8
0.2
0.0
x(k)
0
20
k
60
80


R = 3.5
0.383
0.500
0.827
0.875
0.3
f(x)
x
0.0
0.2
0.8
1.0

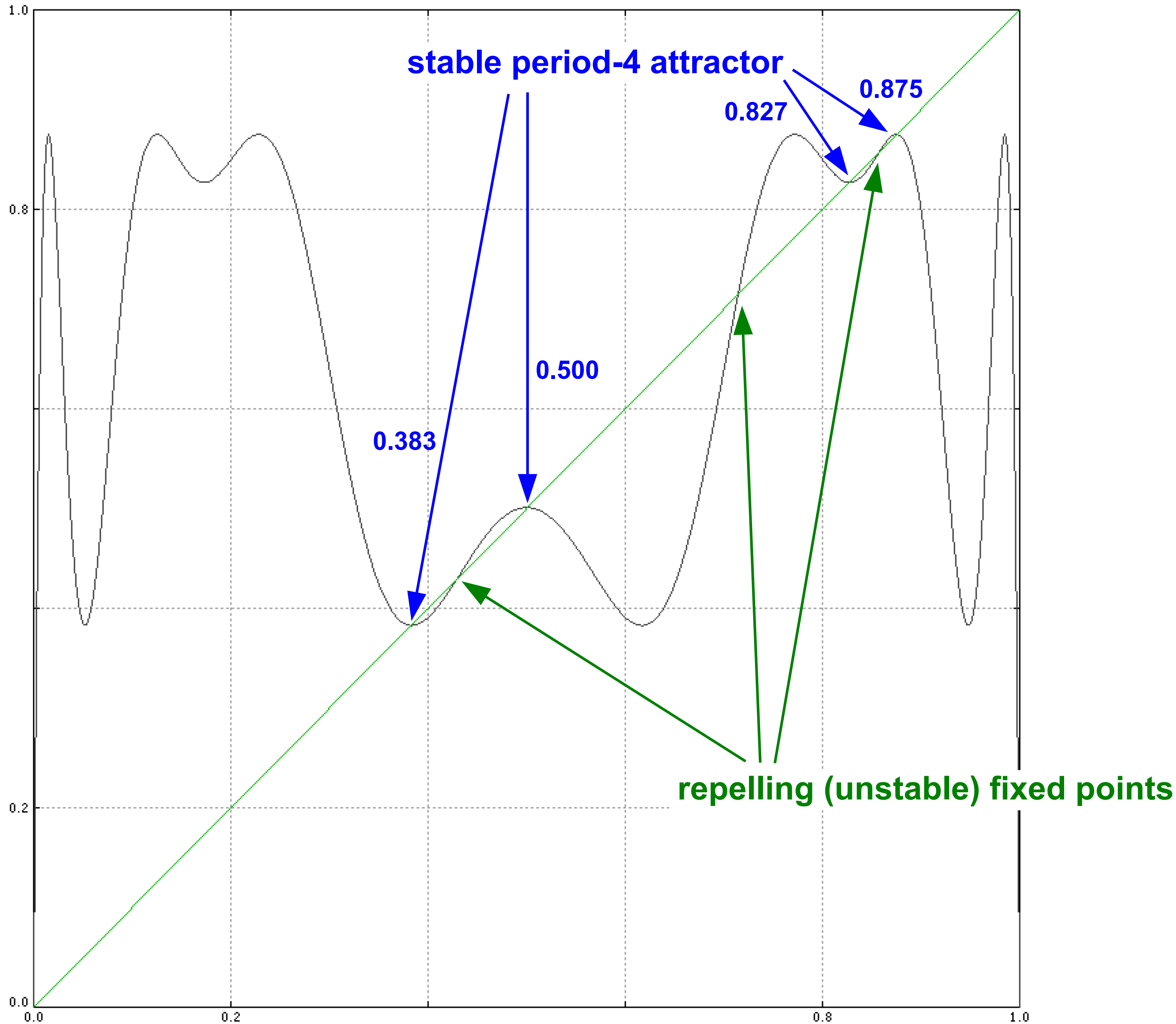


R = 3.5
1.0
0.8
0.2
0.0
f(x)
0.0
0.2
0.8
1.0
x
0.383
0.500
0.827
0.875
0.3

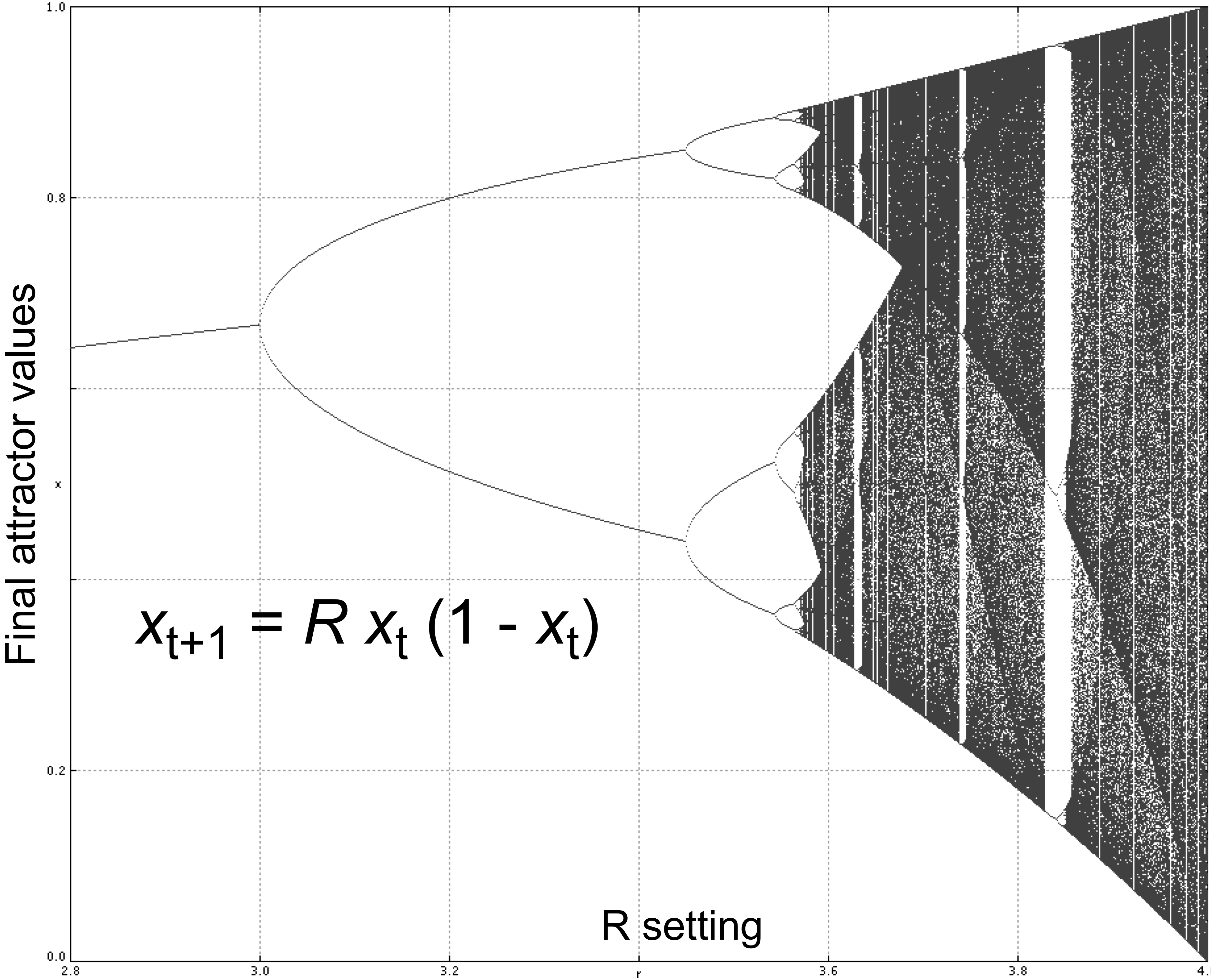
R = 3.5

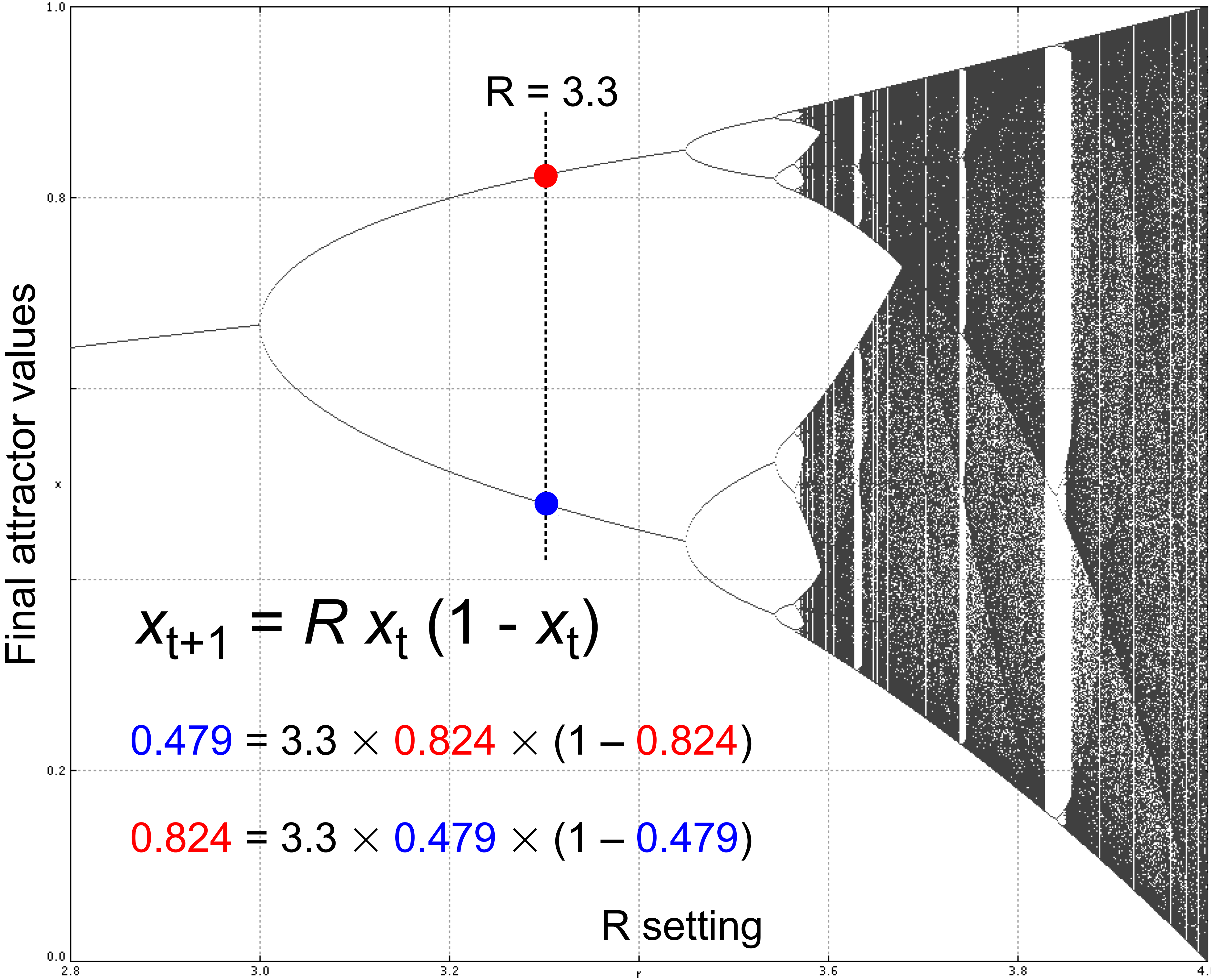


R = 3.5

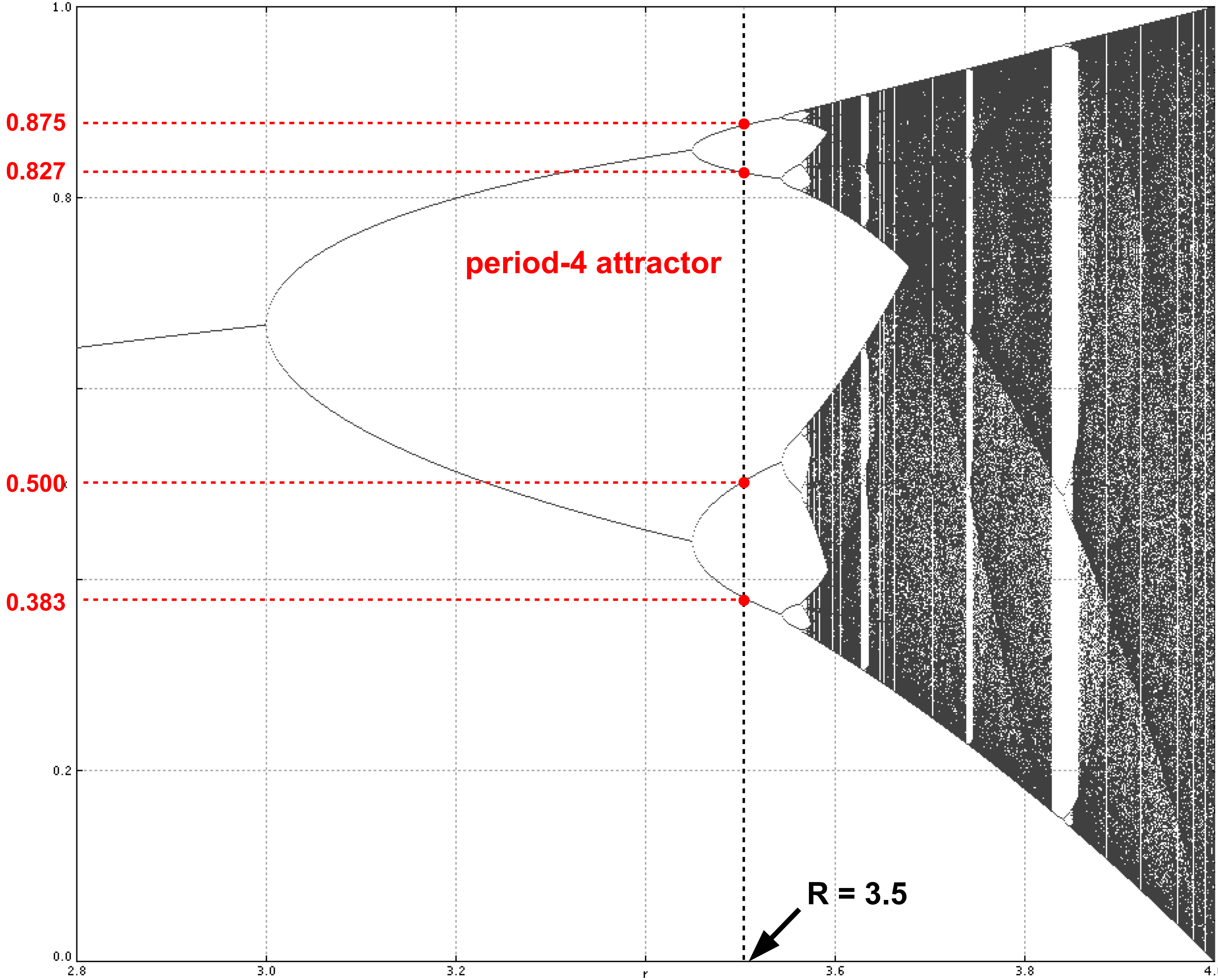


stable period-4 attractor
0.875
0.827
0.500
0.383
repelling (unstable) fixed points
1.0
0.8
0.2
0.0
0.0
0.2
0.8
1.0

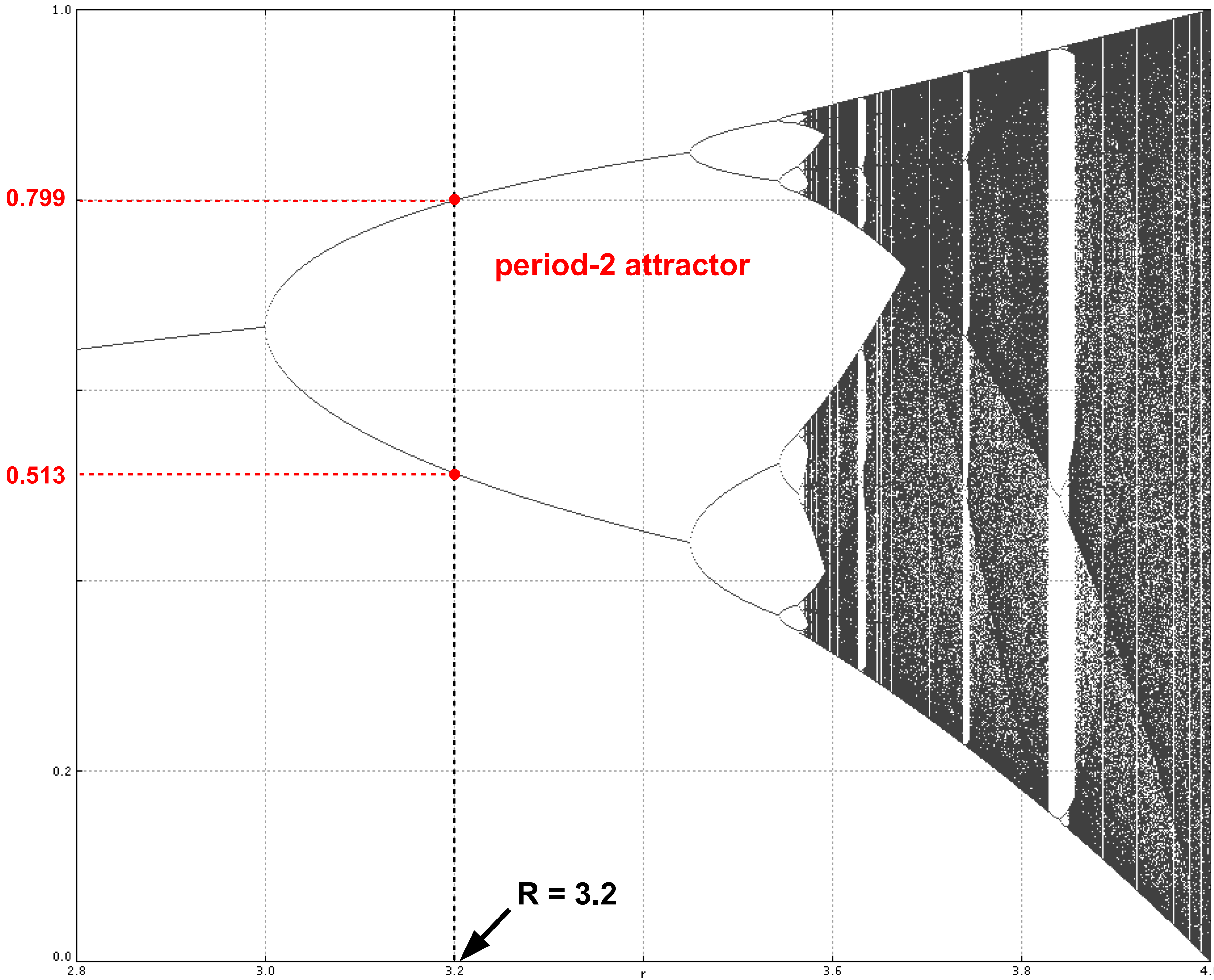

Final attractor values
$x_{t+1} = R\, x_t\, (1 - x_t)$
R setting
1.0
0.8
0.2
0.0
x
2.8
3.0
3.2
r
3.6
3.8
4.0



Final attractor values
R = 3.3
$x_{t+1} = R\, x_t\, (1 - x_t)$
$0.479 = 3.3 \times 0.824 \times (1 - 0.824)$
$0.824 = 3.3 \times 0.479 \times (1 - 0.479)$
R setting
1.0
0.8
x
0.2
0.0
2.8
3.0
3.2
r
3.6
3.8
4.0

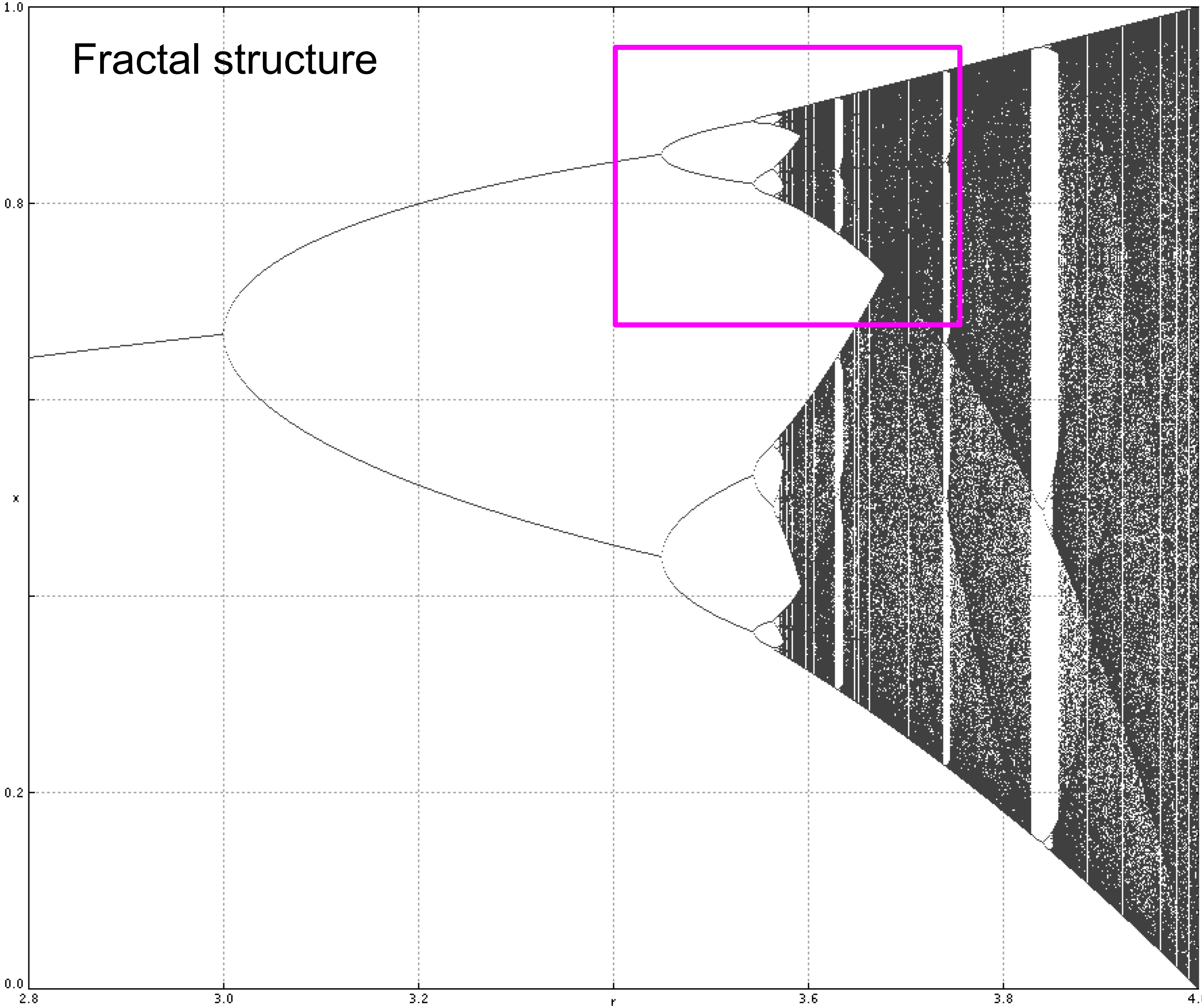


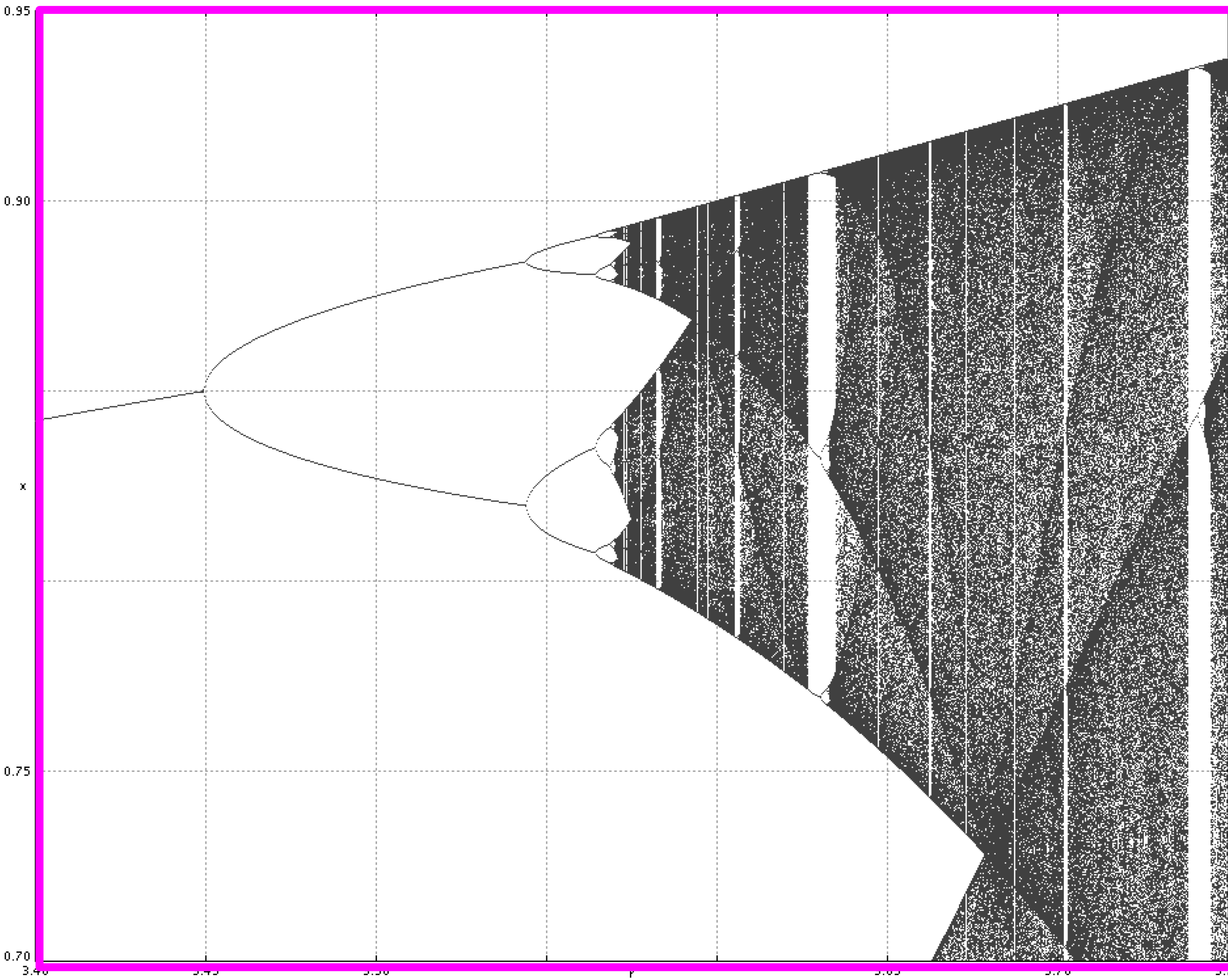
1.0
0.8
0.2
0.0
2.8
3.0
3.2
r
3.6
3.8
4.0
0.875
0.827
0.500
0.383
period-4 attractor
R = 3.5

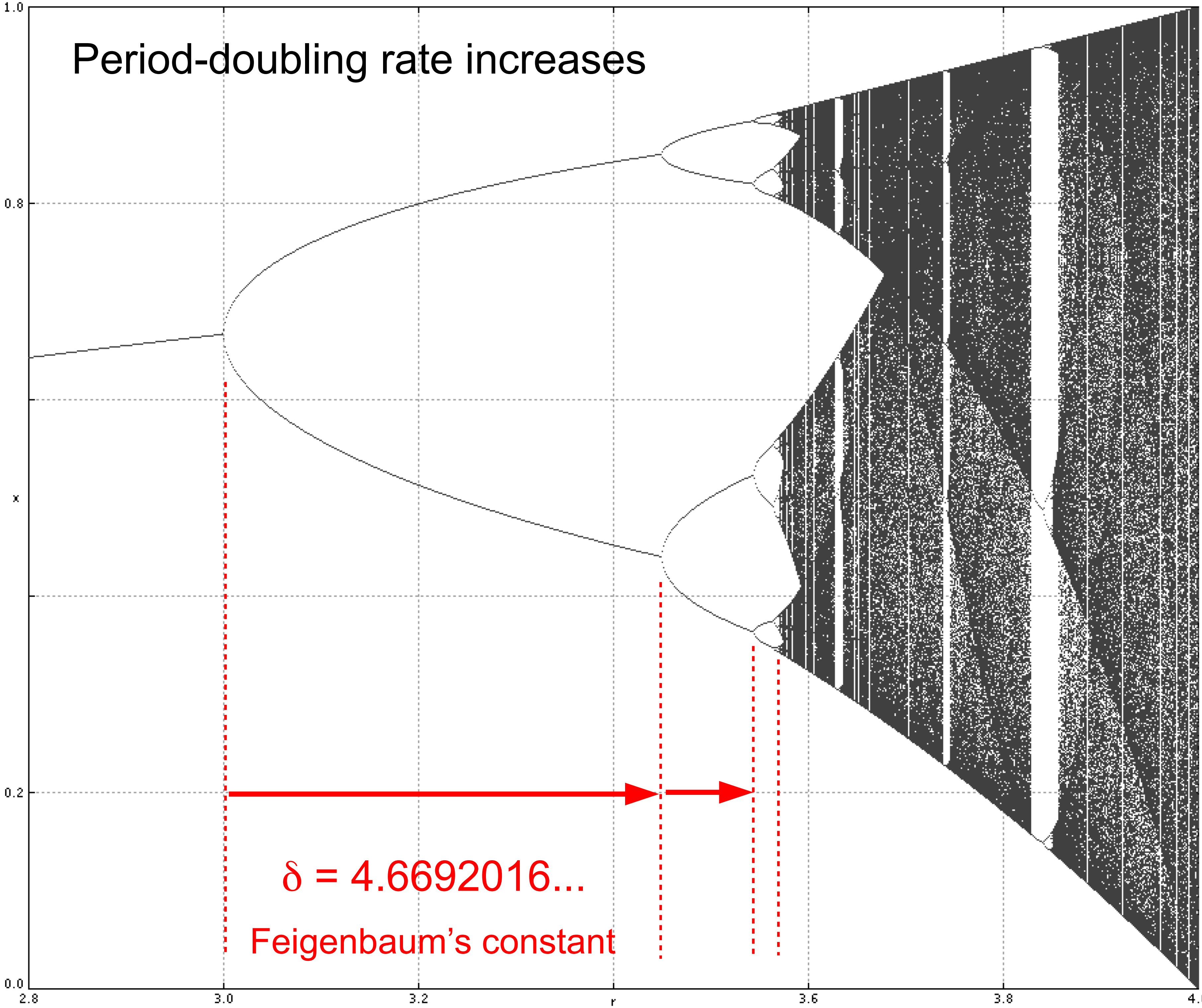


1.0
0.799
0.513
0.2
0.0
2.8
3.0
3.2
r
3.6
3.8
4.0
period-2 attractor
R = 3.2

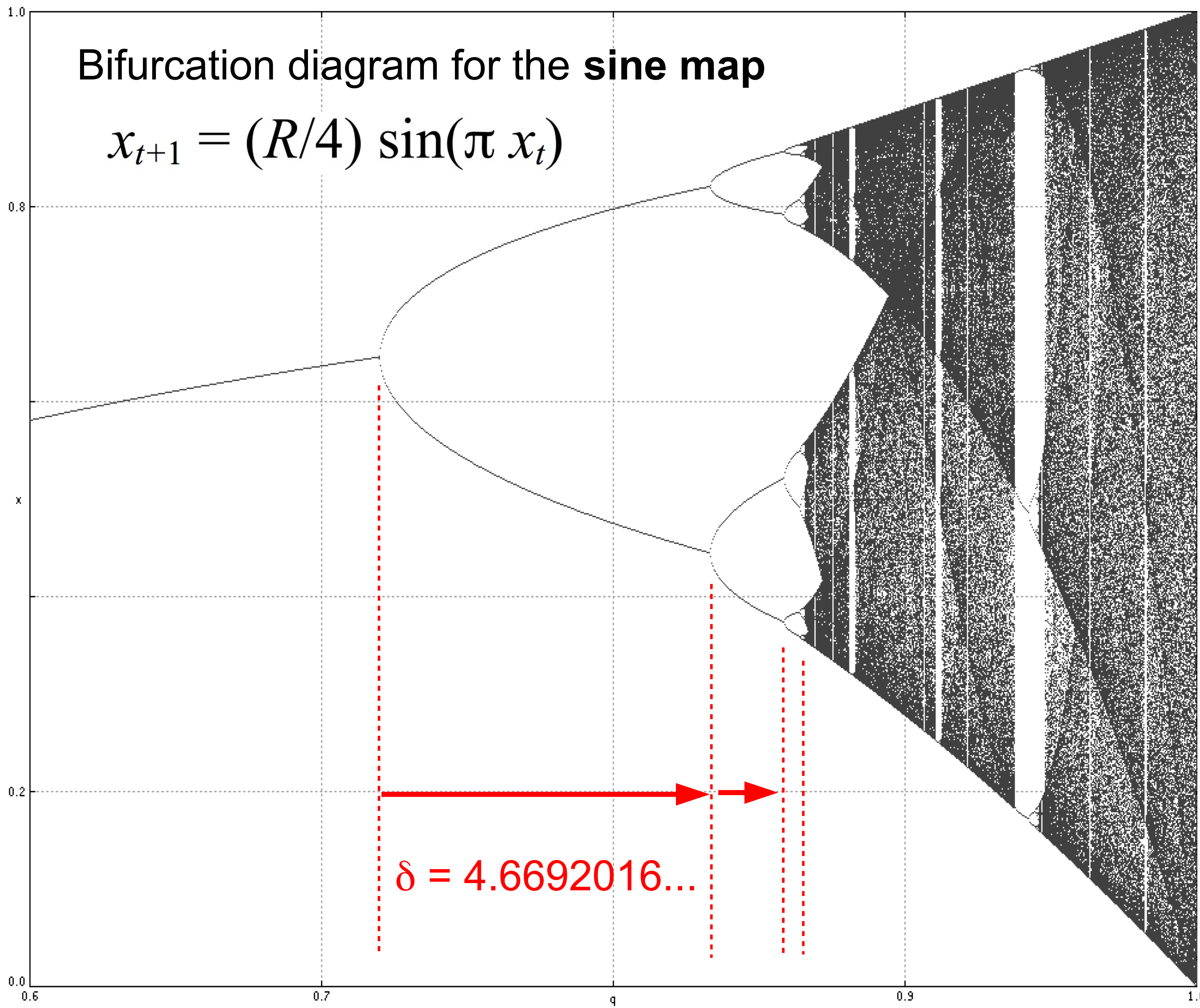
Fractal structure




0.95
0.90
x
0.75
0.70
p



Period-doubling rate increases
δ = 4.6692016...
Feigenbaum's constant
1.0
0.8
0.2
0.0
x
2.8
3.0
3.2
r
3.6
3.8
4.0


Bifurcation diagram for the **sine map**
$x_{t+1} = (R/4)\sin(\pi x_t)$
$\delta = 4.6692016...$
1.0
0.8
0.2
0.0
x
0.6
0.7
q
0.9
1.0

Bifurcation Diagram Zoom