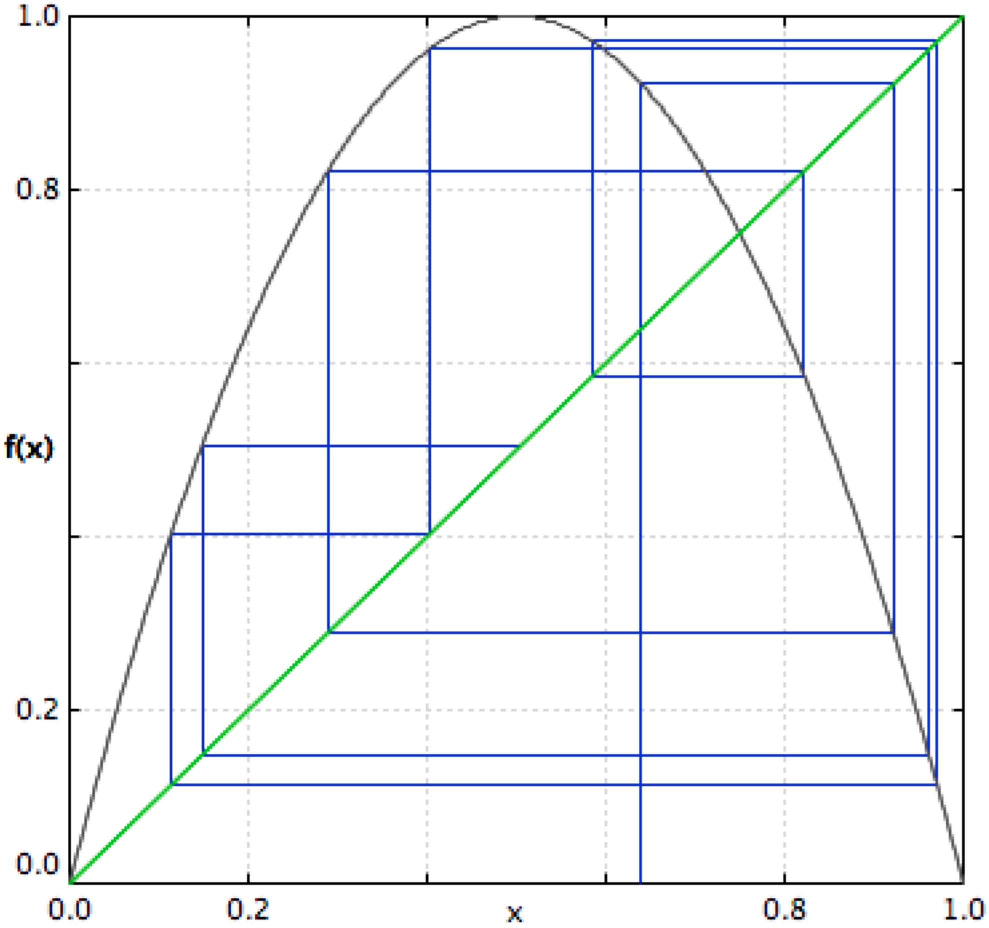
Strange Attractors

Reading

CBN Chapter 11: sections 11.1, 11.3

State Space

- Logistic Map with R = 4.0
- A chaotic trajectory fills the entire **1-dimensional state space**

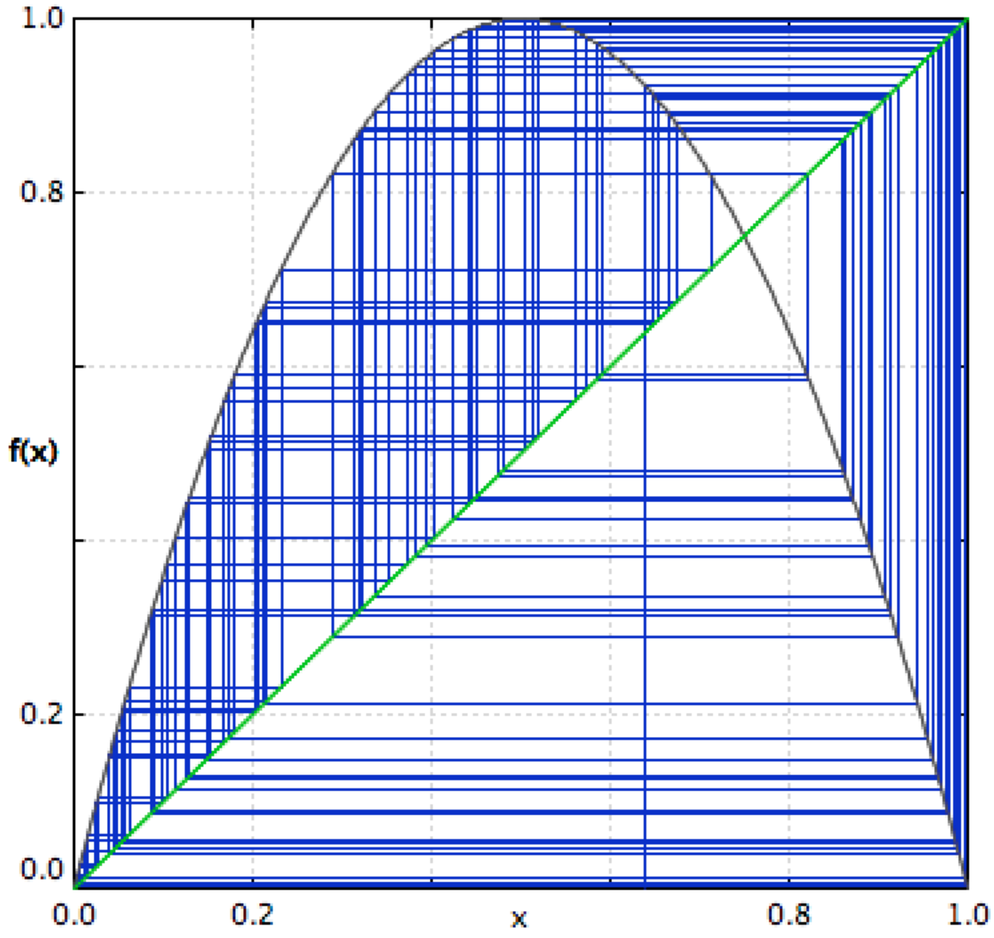


$x_0 = 0.64$

10 iterations

State Space

- Logistic Map with R = 4.0
- A chaotic trajectory fills the entire **1-dimensional state space**

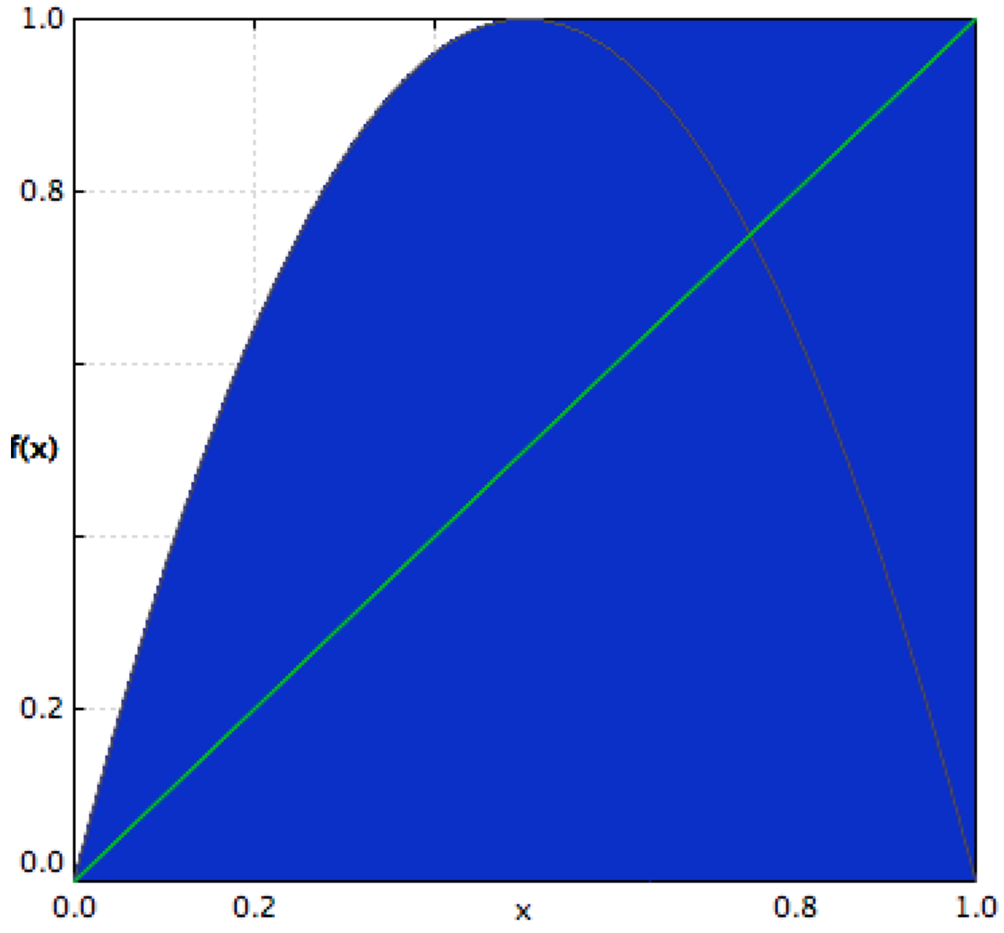


$x_0 = 0.64$

100 iterations

State Space

- Logistic Map with R = 4.0
- A chaotic trajectory fills the entire **1-dimensional state space**



$x_0 = 0.64$

10000 iterations

State Spaces

- We need to consider state spaces with **more than 1 dimension**
  - Stationary object on a flat surface (2-D)
  - Moving ball on a flat surface (4-D)
  - Earth + Moon + satellite system (18-D)
- **State variables** summarize all relevant information about the entire system (position, velocity, etc. of each component)
- Together the state variables represent a **single abstract point** in a multi-dimensional state space

The Lorenz Equations

- Studied by **Edward Lorenz** in 1963 as a simple model of weather

$$x' = Ay - Ax$$

$$y' = Bx - y - zx$$

$$z' = xy - Cz$$

- Idealized model of **convective fluid motion** in the atmosphere
- A, B, C are constants that reflect **physical properties** of the fluid
- System is **chaotic** when $A = 10$, $B = 28$, $C = 8/3\ (2.6667)$
- System exhibits **sensitive dependence** on initial conditions

The Lorenz Equations

$$x' = Ay - Ax$$

$$y' = Bx - y - zx$$

$$z' = xy - Cz$$

- x, y, and z are the **state variables**

  x is proportional to the intensity of convective motion

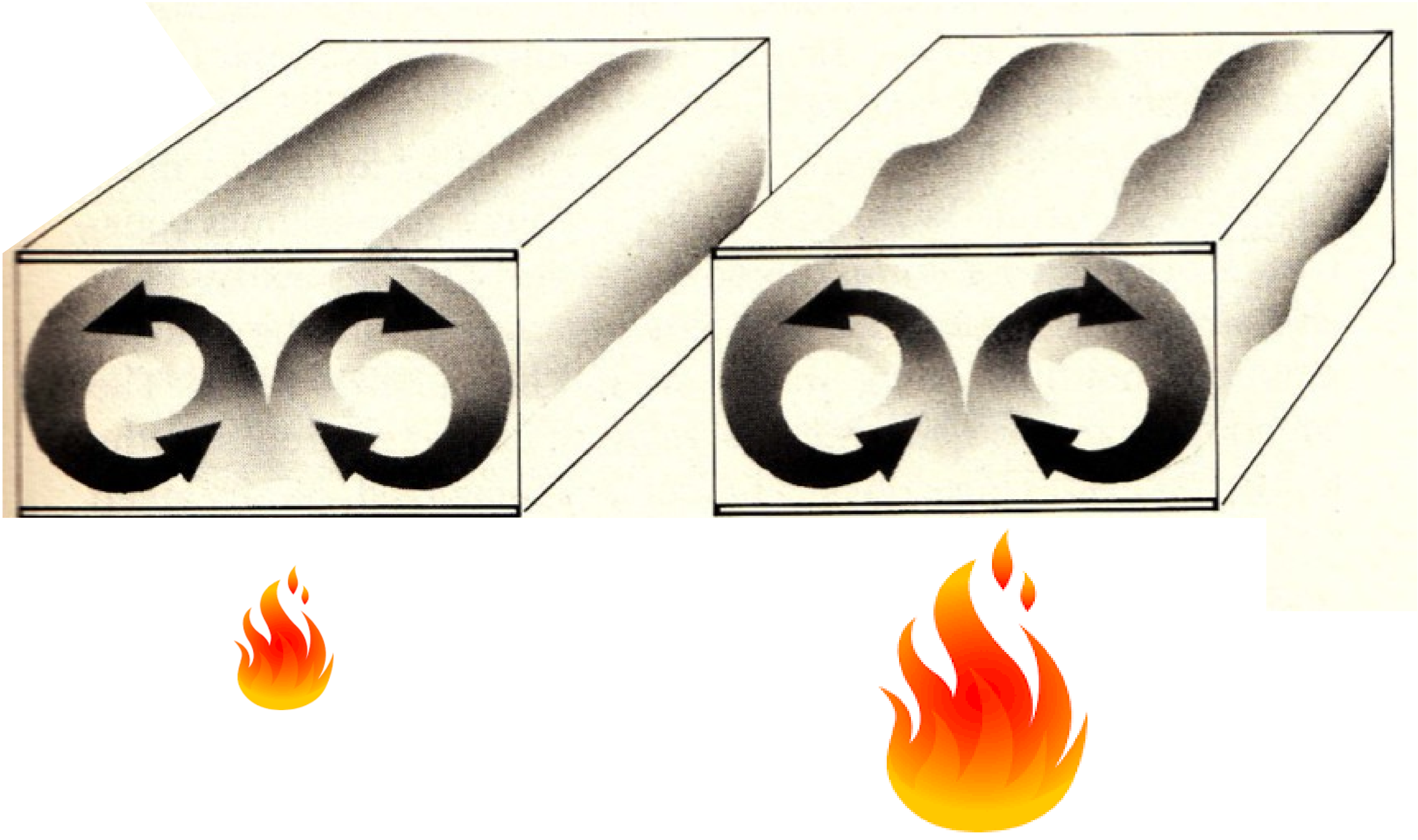
  y is proportional to the horizontal temperature gradient between ascending and descending air currents

  z is proportional to the vertical temperature gradient

- x', y', and z' are the **rates** at which x, y, and z are **changing**

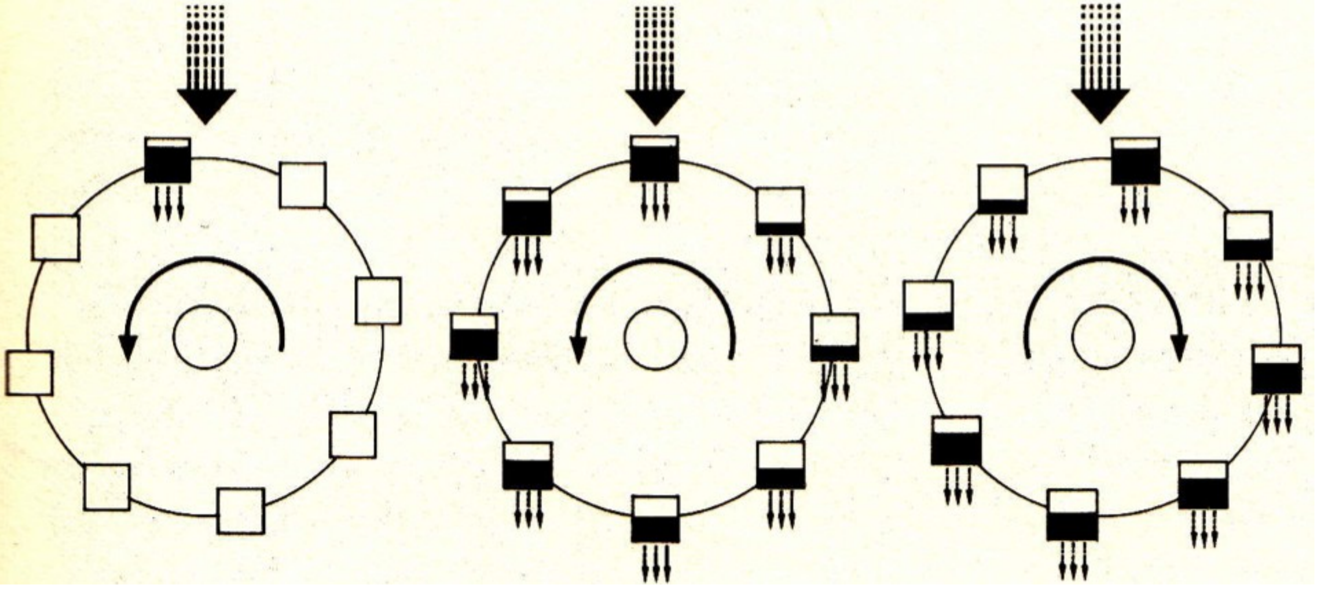
Convective Fluid Motion

- The rising of hot gas or liquid
- Nonlinearities due to friction and viscosity



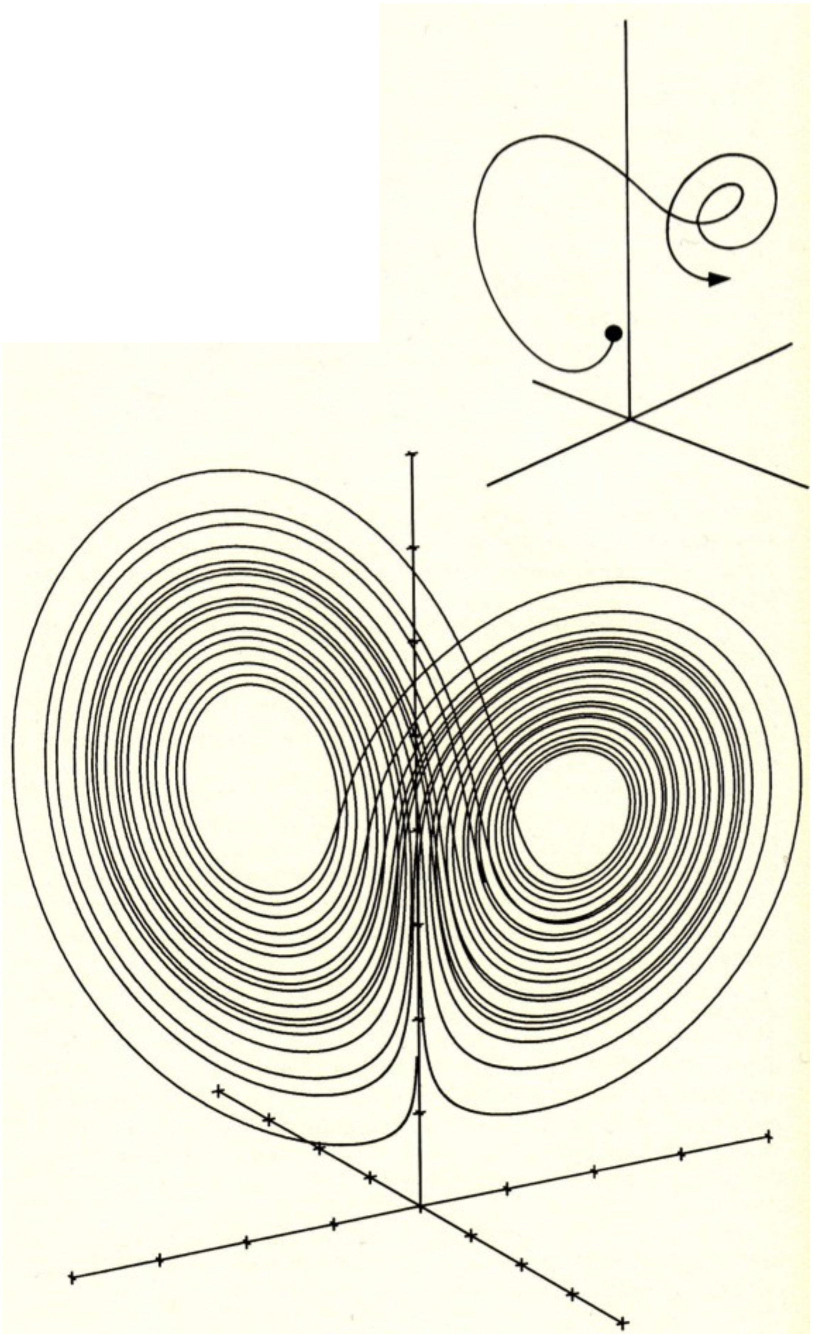
The Lorenz Waterwheel

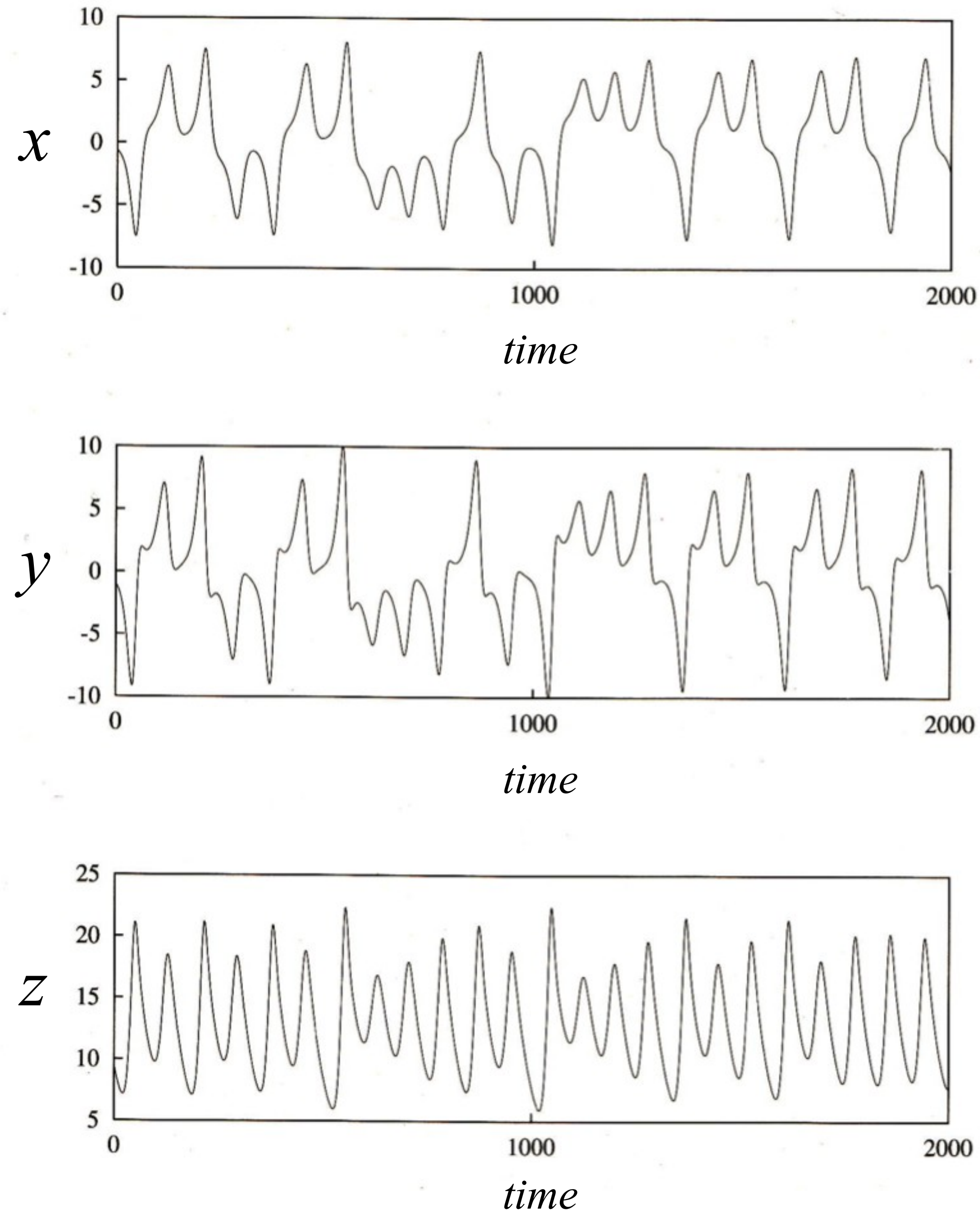
- The Lorenz equations correspond exactly to a real mechanical device: a waterwheel
- Like a slice through a rotating convection cylinder
- Both systems have an external energy source (heat / water)
- Both systems dissipate energy
- Water flows in from top, and buckets leak water at a fixed rate

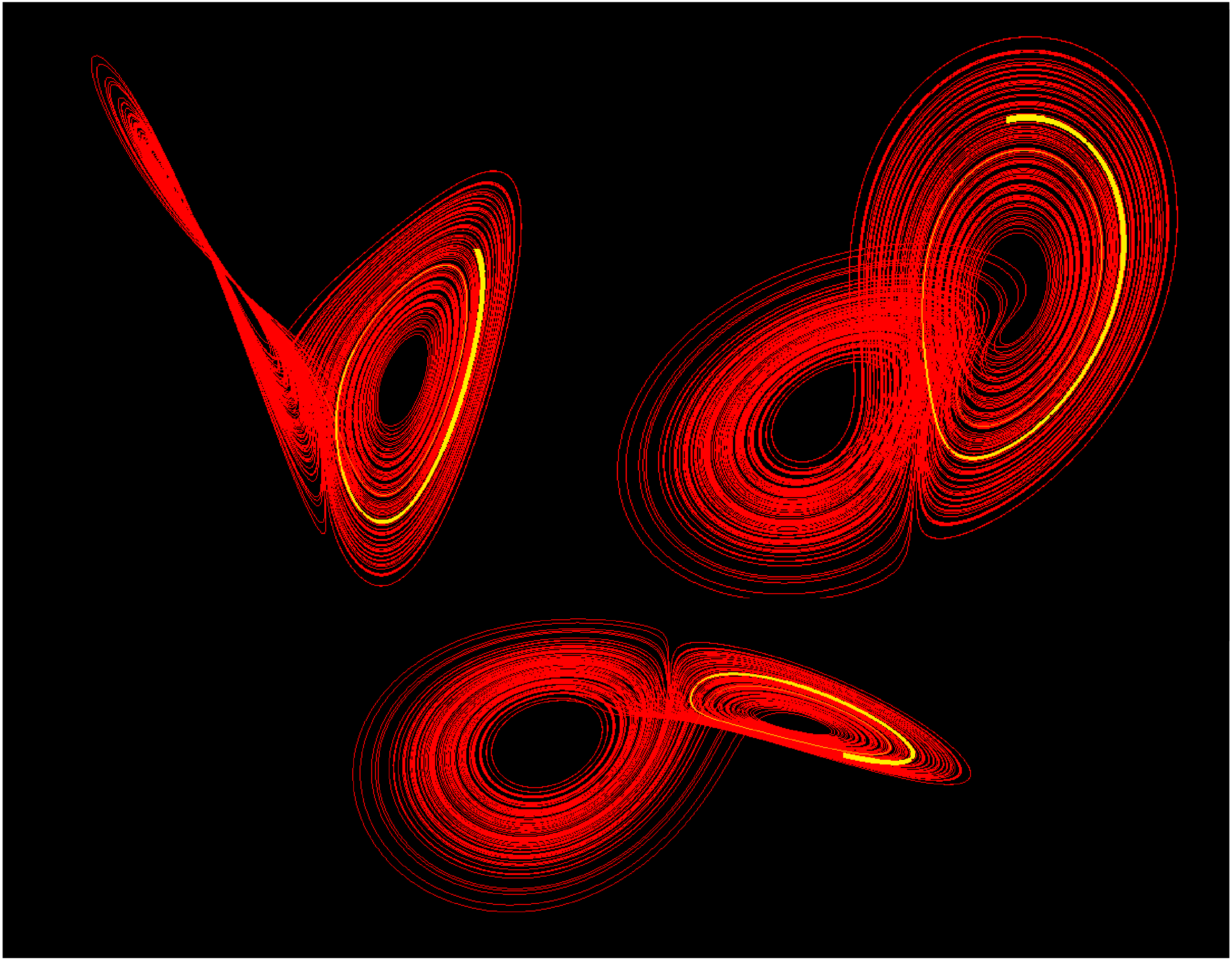


The Lorenz Waterwheel

The Lorenz Attractor



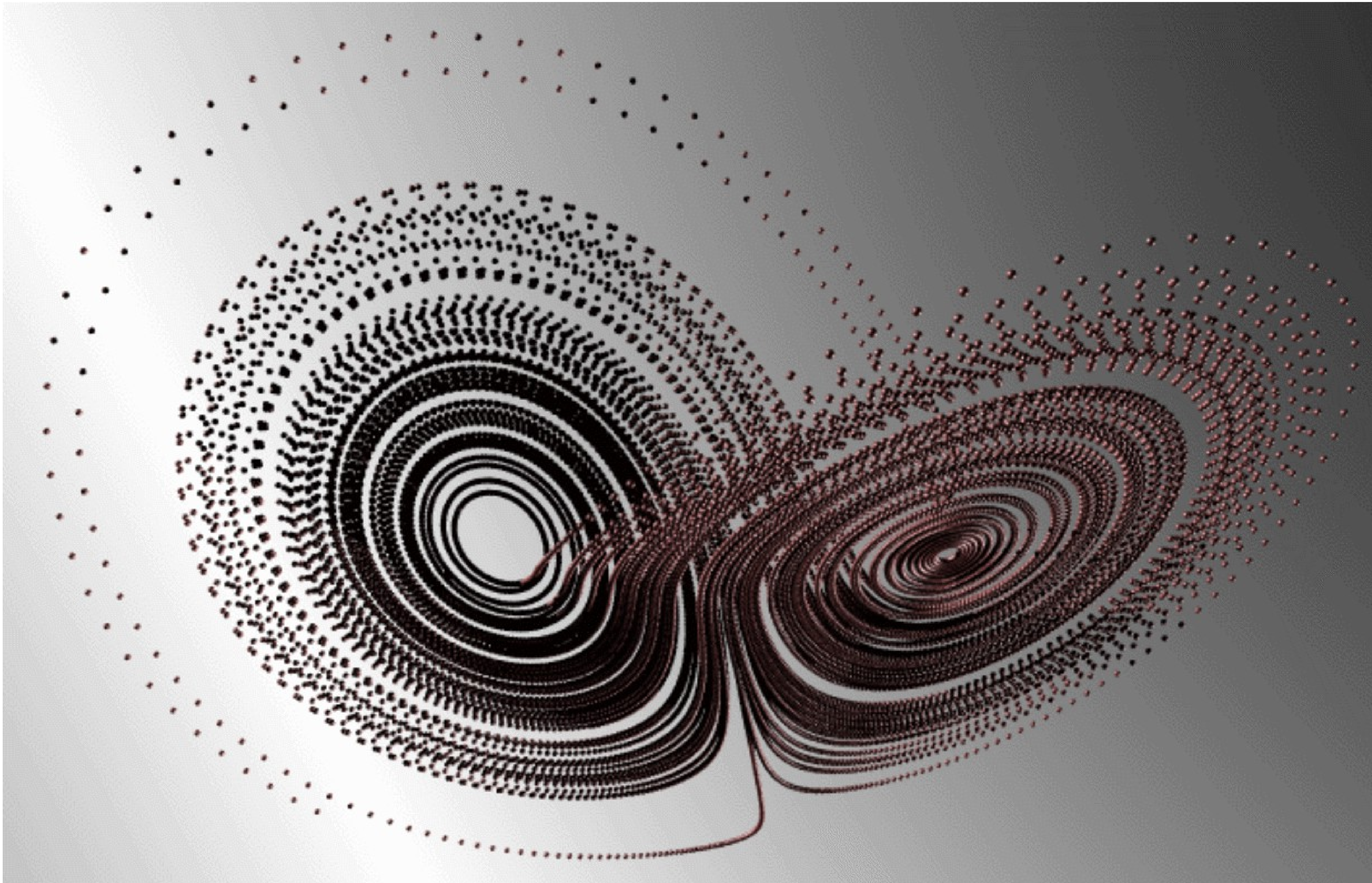




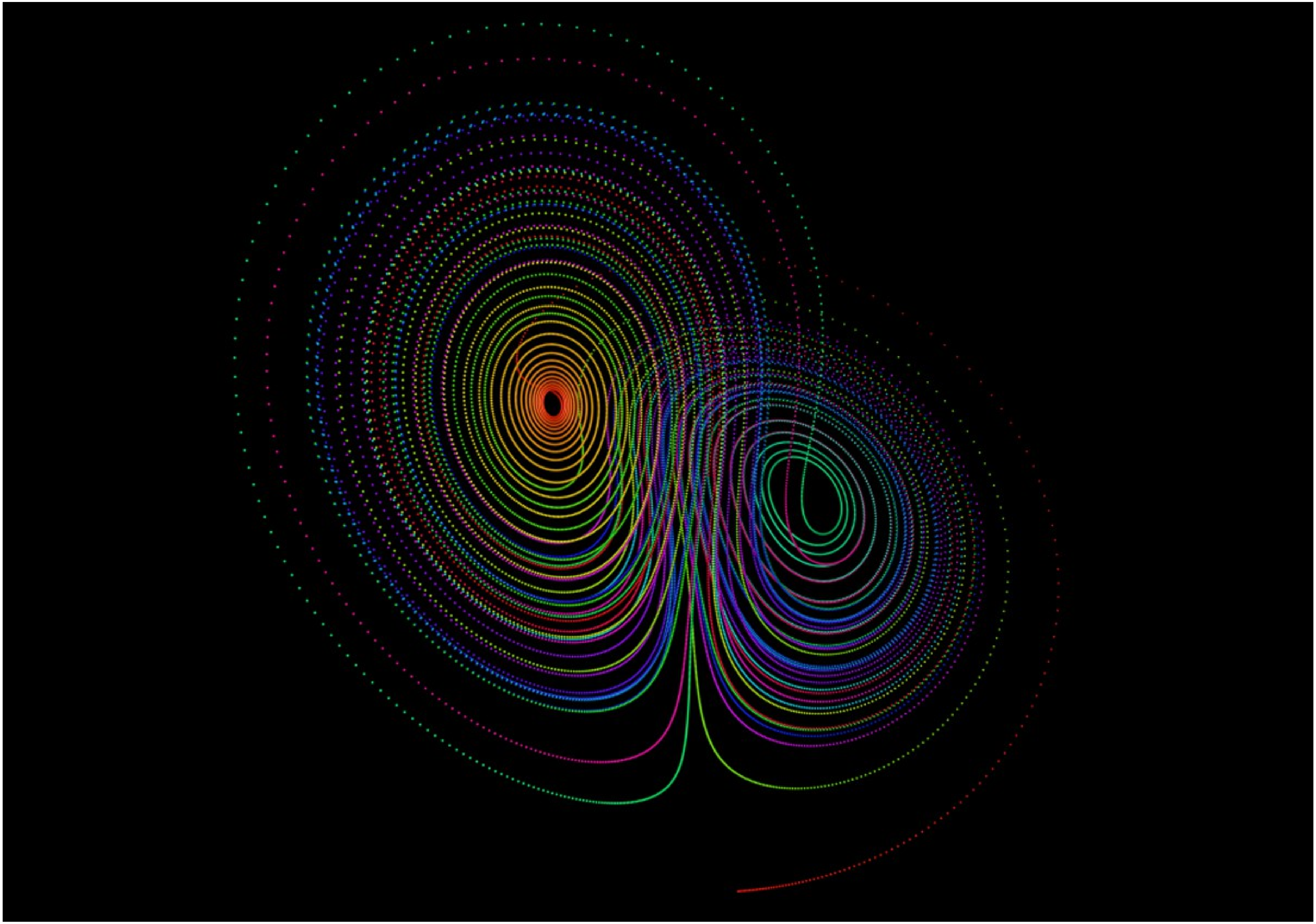
The Lorenz Attractor

- Long-term behavior of the system in state space is confined to the surface of the “butterfly”
- An example of a “strange attractor” in 3-D
- State trajectory never intersects itself (it is infinitely dense)
- How long the trajectory stays on each “wing” is unpredictable
- In 1999, Warwick Tucker in Sweden proved mathematically that the Lorenz attractor really exists!

Images of the Lorenz Attractor by Paul Bourke



Images of the Lorenz Attractor by Paul Bourke



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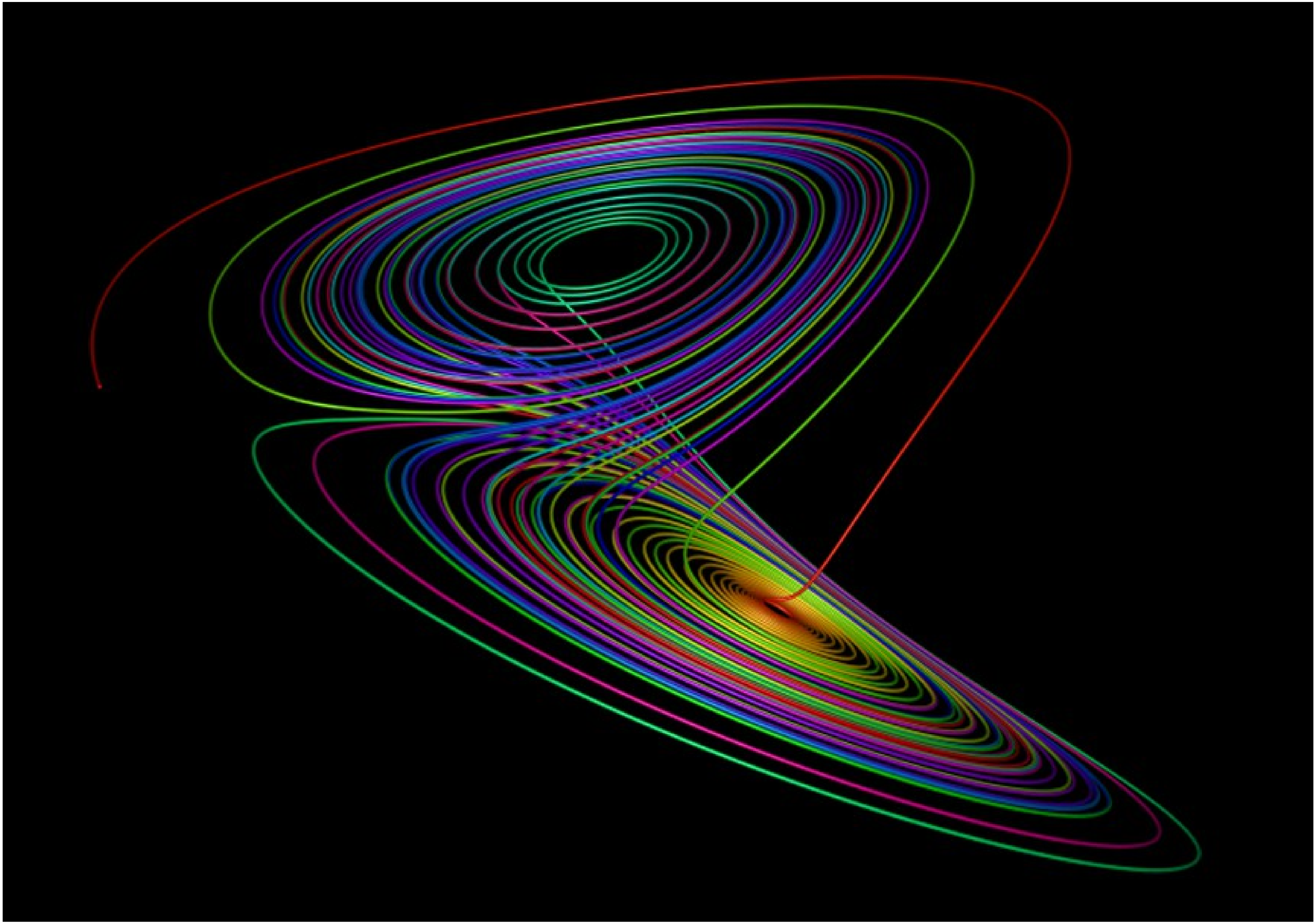

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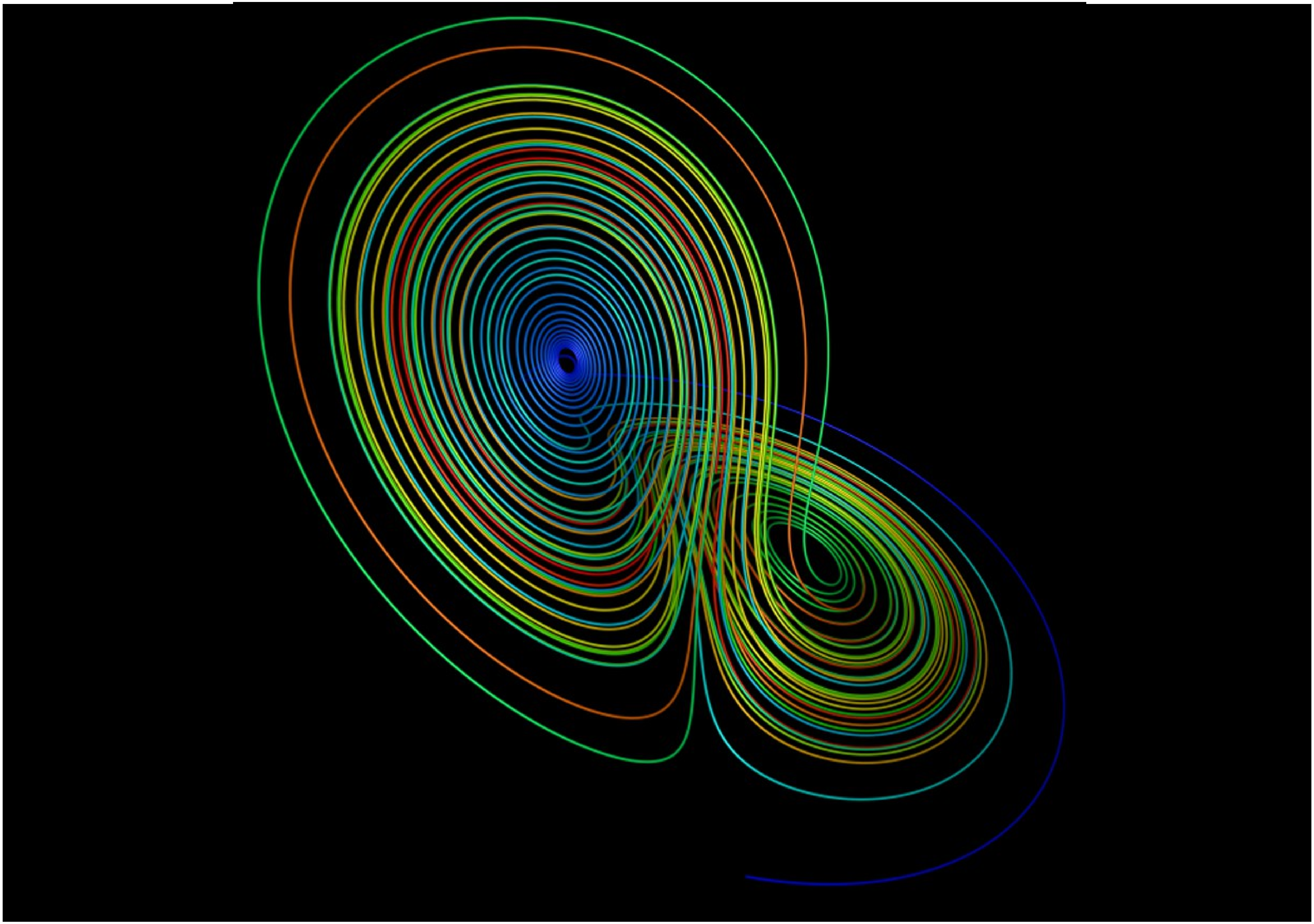

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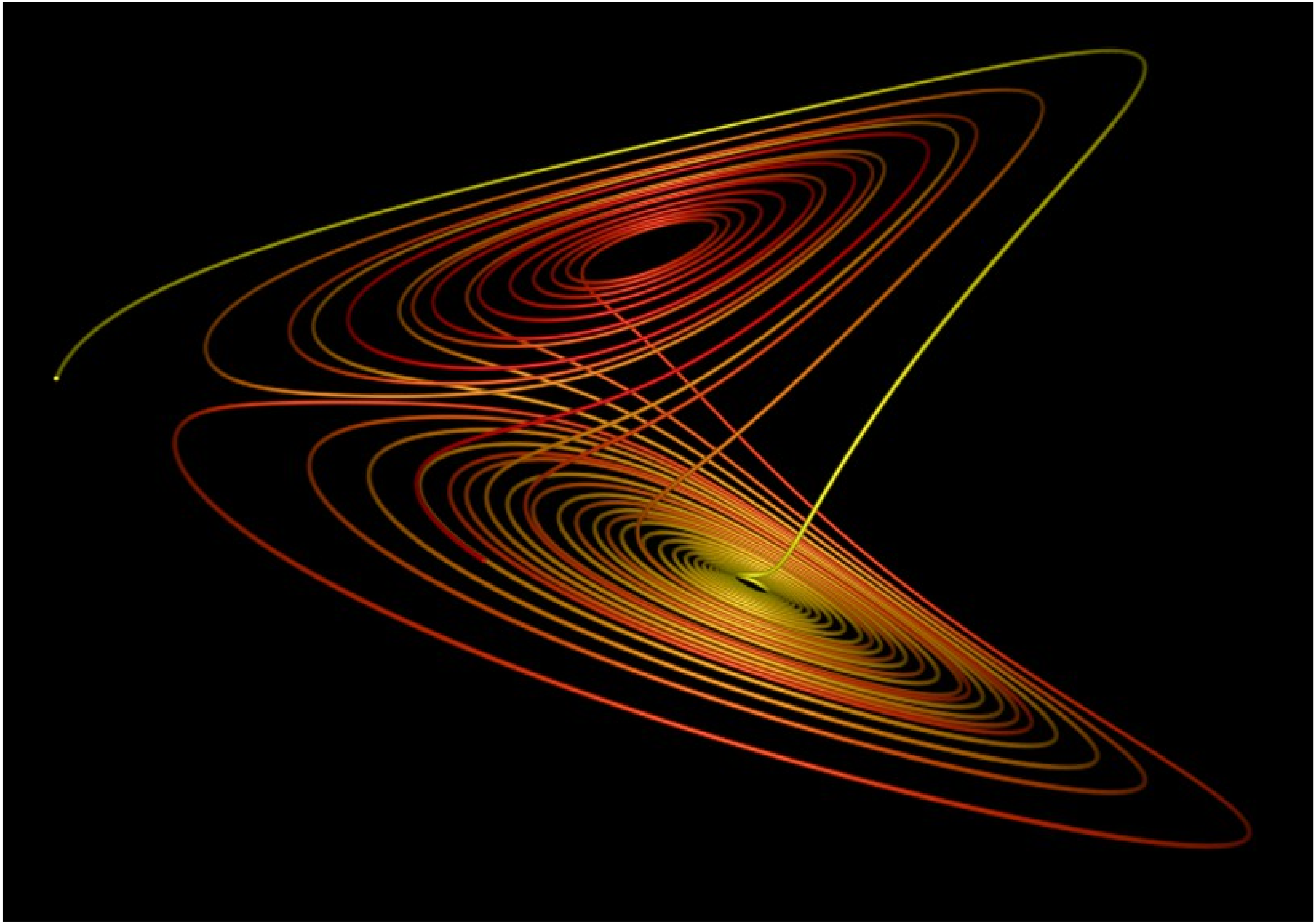

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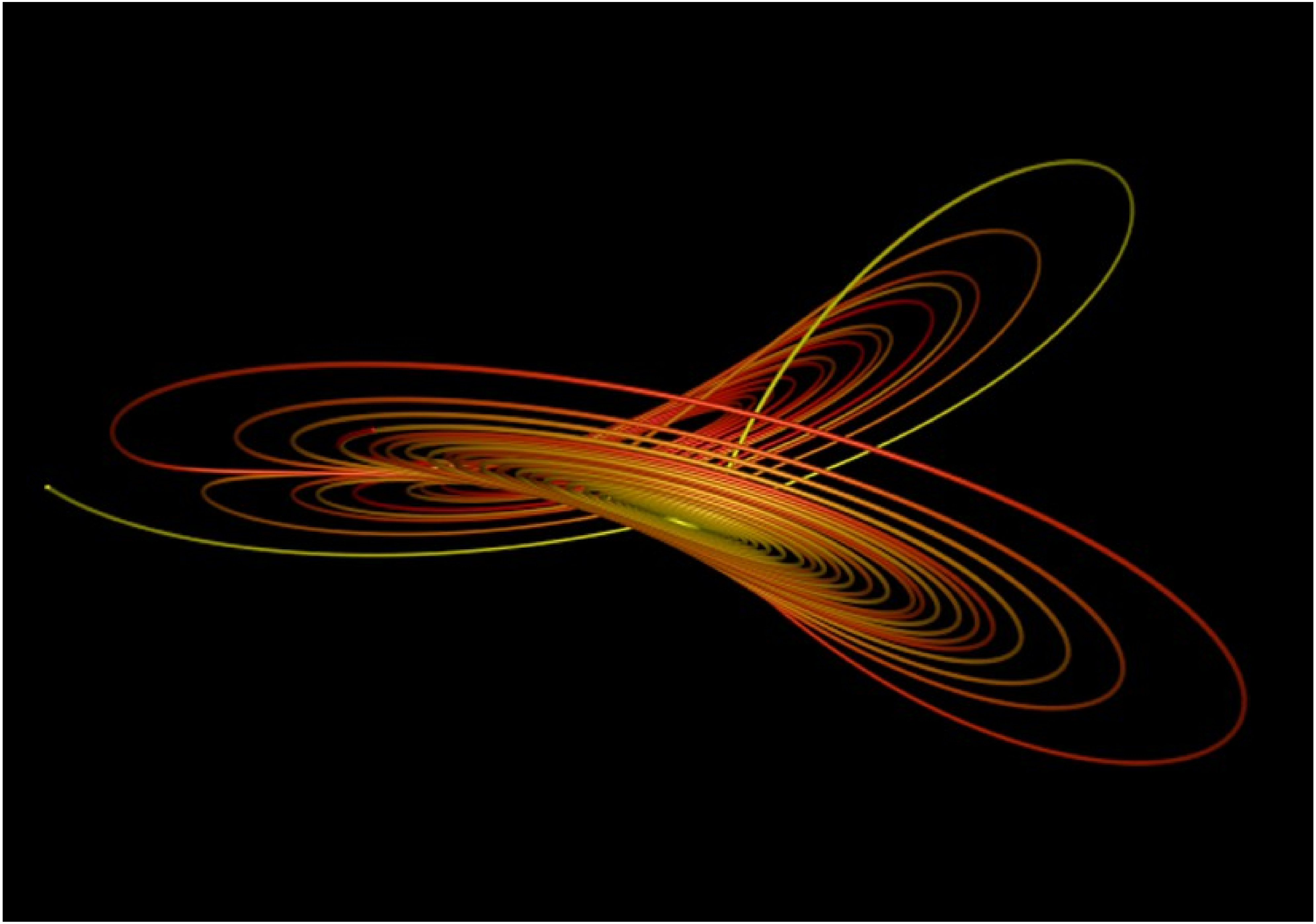

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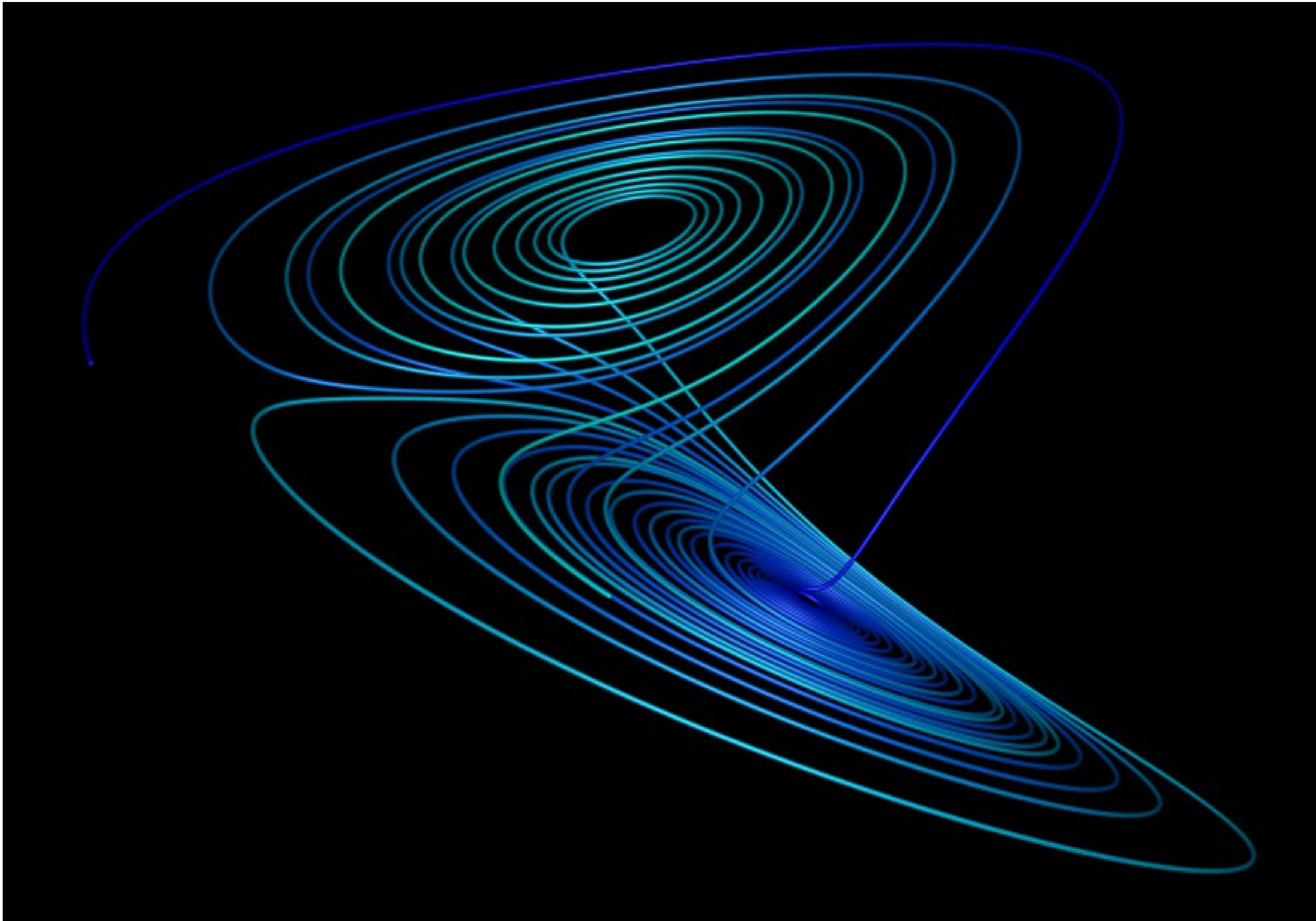

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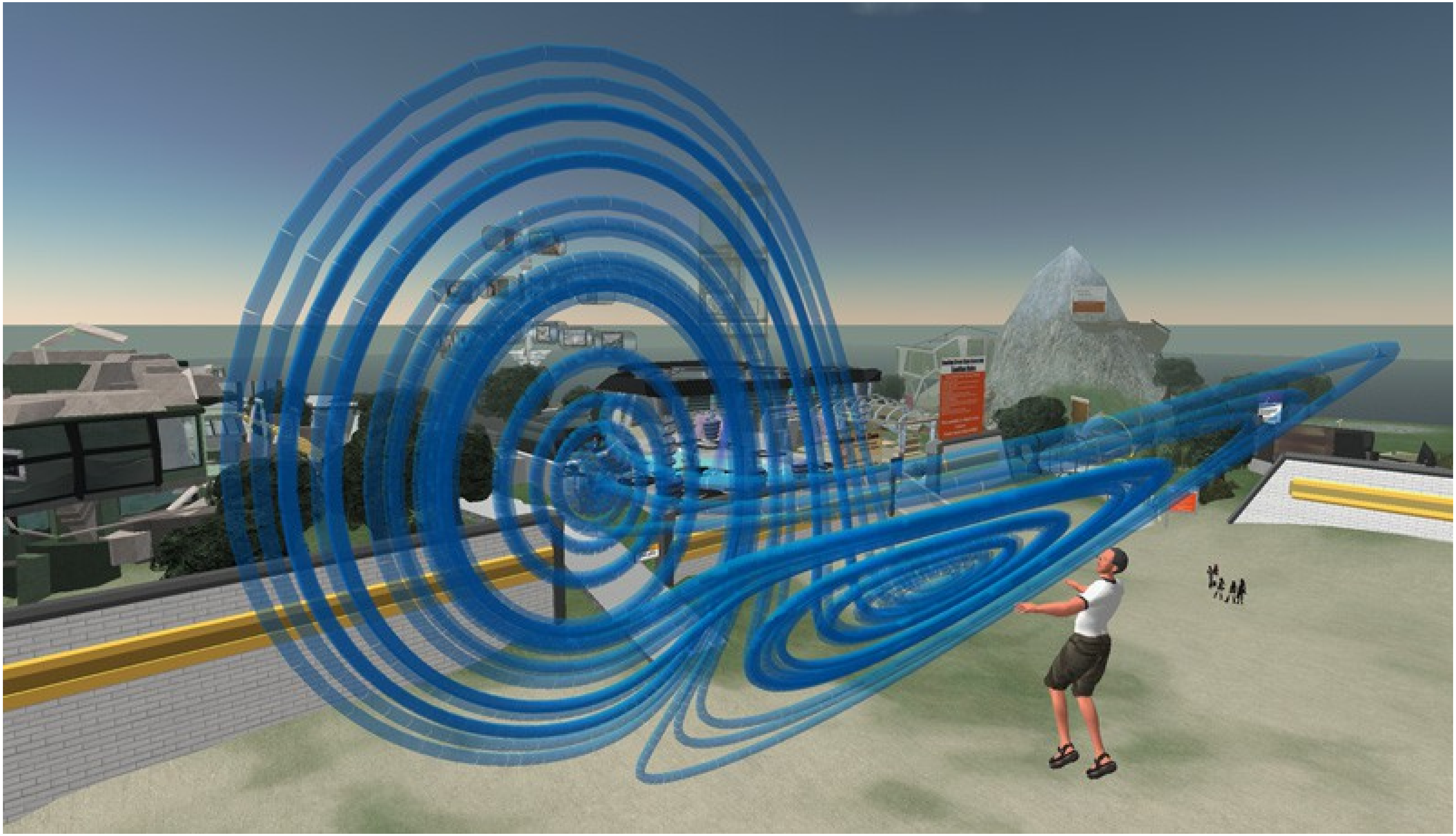
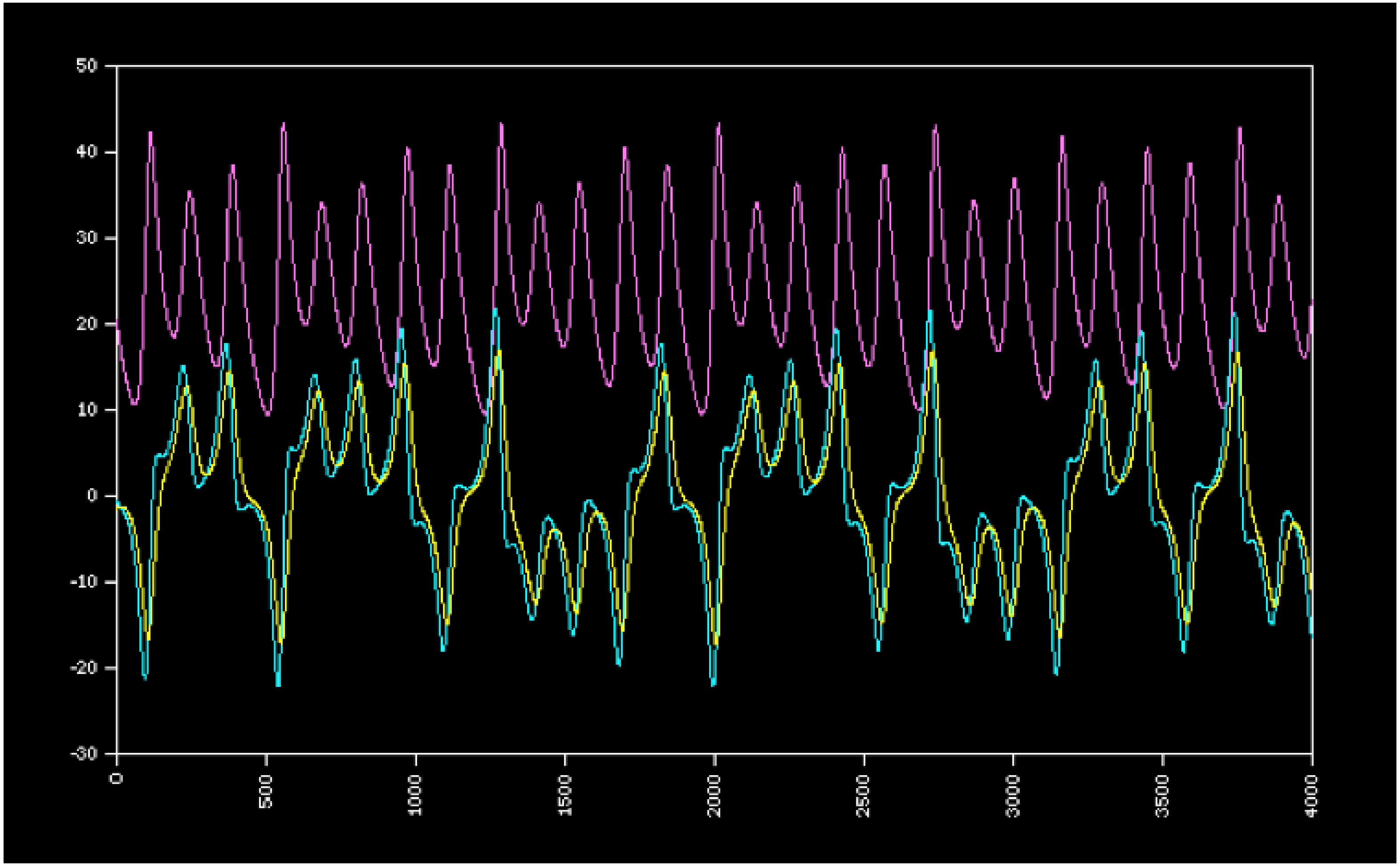


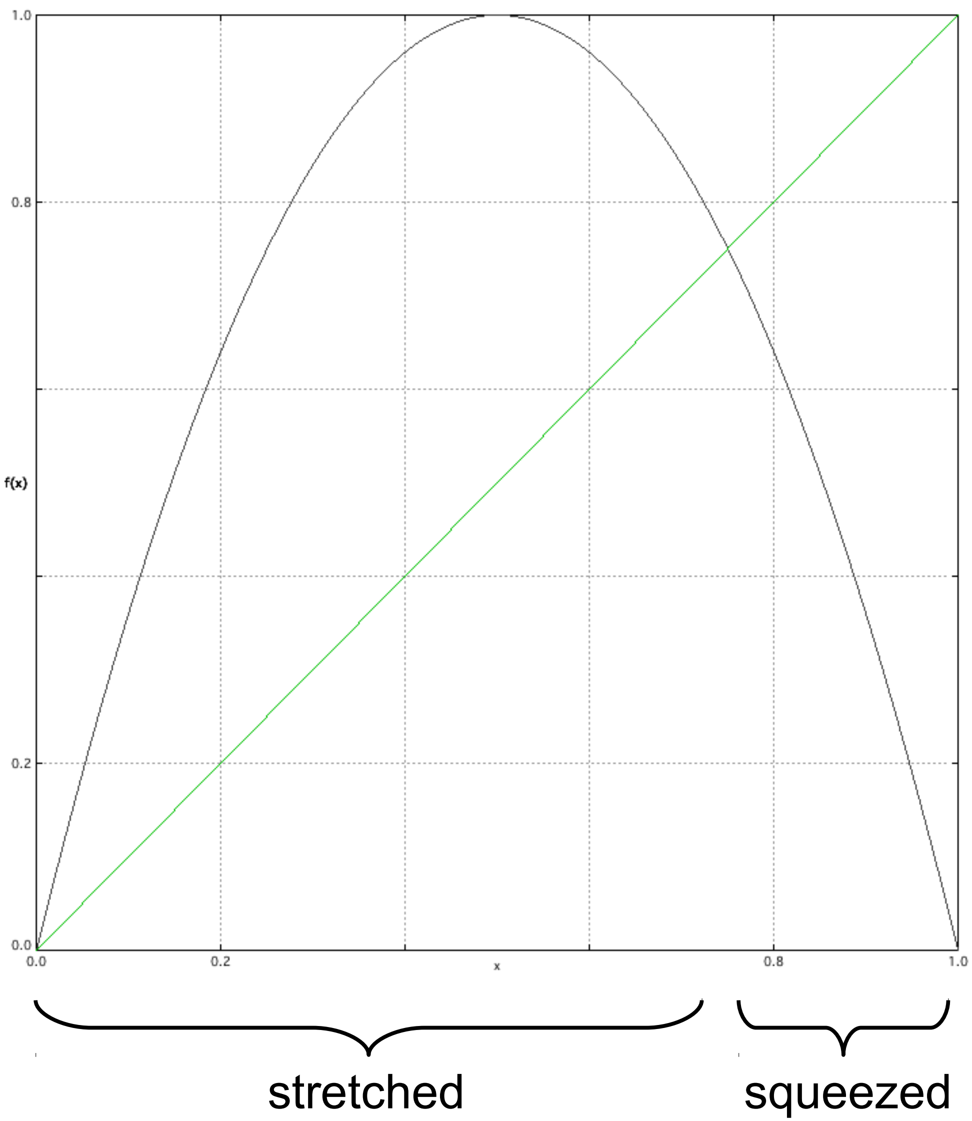
The Sound of the Lorenz Attractor

Each axis (x, y, and z) is mapped to a different instrument



The Hénon Attractor

The Logistic Map



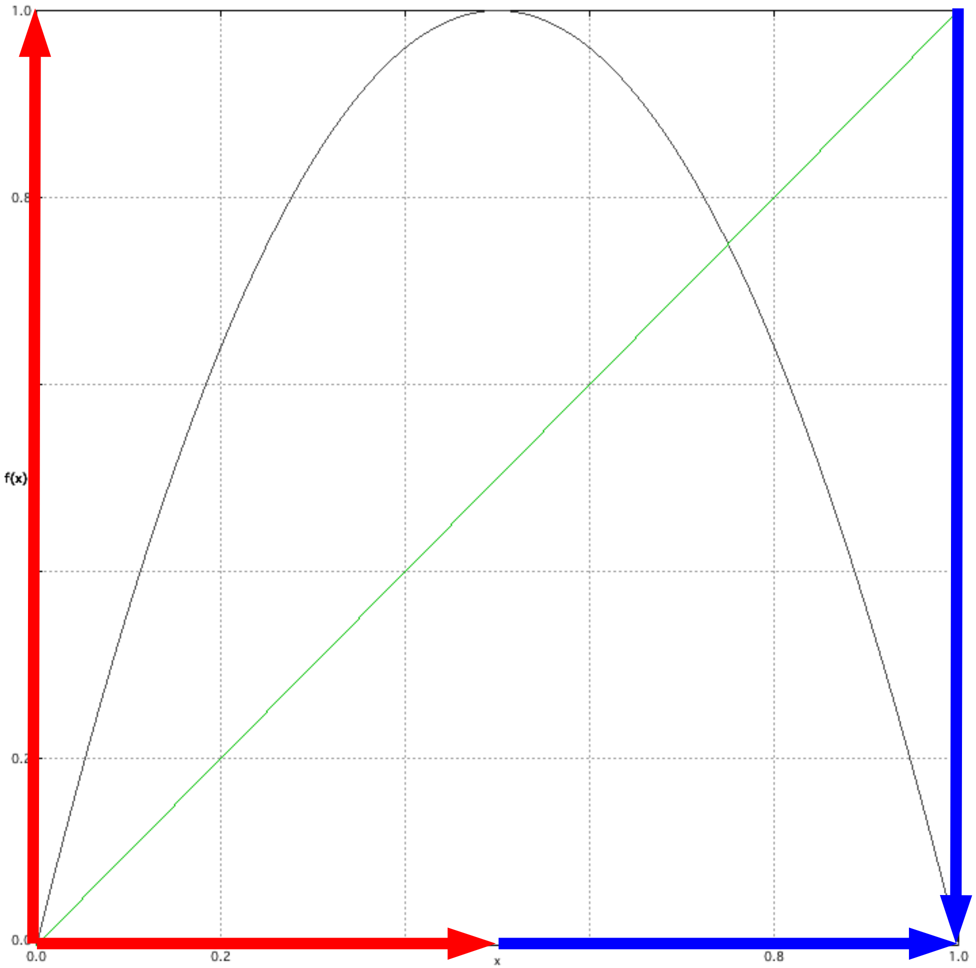
Smaller values of x get “stretched”

$0.2 \rightarrow 0.64$
$0.4 \rightarrow 0.96$
$0.5 \rightarrow 1.0$
$0.6 \rightarrow 0.96$

Larger values of x get “squeezed”

$0.8 \rightarrow 0.64$
$0.9 \rightarrow 0.36$
$0.95 \rightarrow 0.19$

The Logistic Map



The space gets “folded”

[0 ... 0.5] → [0 ... 1]

[0.5 ... 1] → [1 ... 0]

Repeated iterations stretch, squeeze, and fold the space, like saltwater taffy, or pastry dough

This is hard to visualize with a 1-dimensional state space

The Henon Map

- 2-dimensional discrete system with state variables x and y

$$x_{t+1} = 1 - Ax_t^2 + y_t$$

$$y_{t+1} = Bx_t$$

- $A = 1.4, \; B = 0.3$ (analogous to the logistic map R parameter)
- The form of the equations given in *CBN* Chapter 11 are slightly different, but mathematically equivalent to the above equations

The Henon Map

- 2-dimensional discrete system with state variables x and y

$$x_{t+1} = 1 - Ax_t^2 + y_t$$

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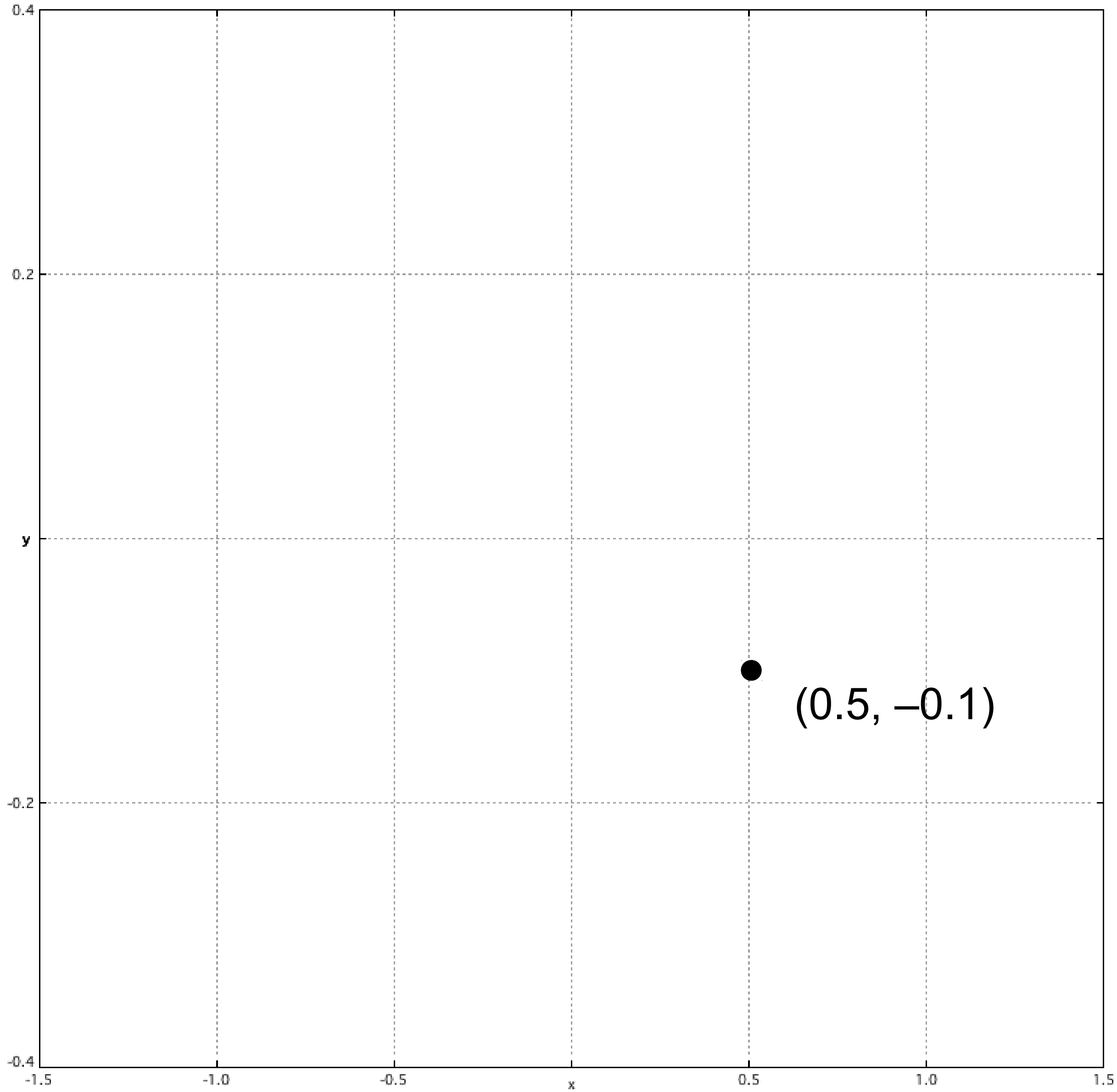
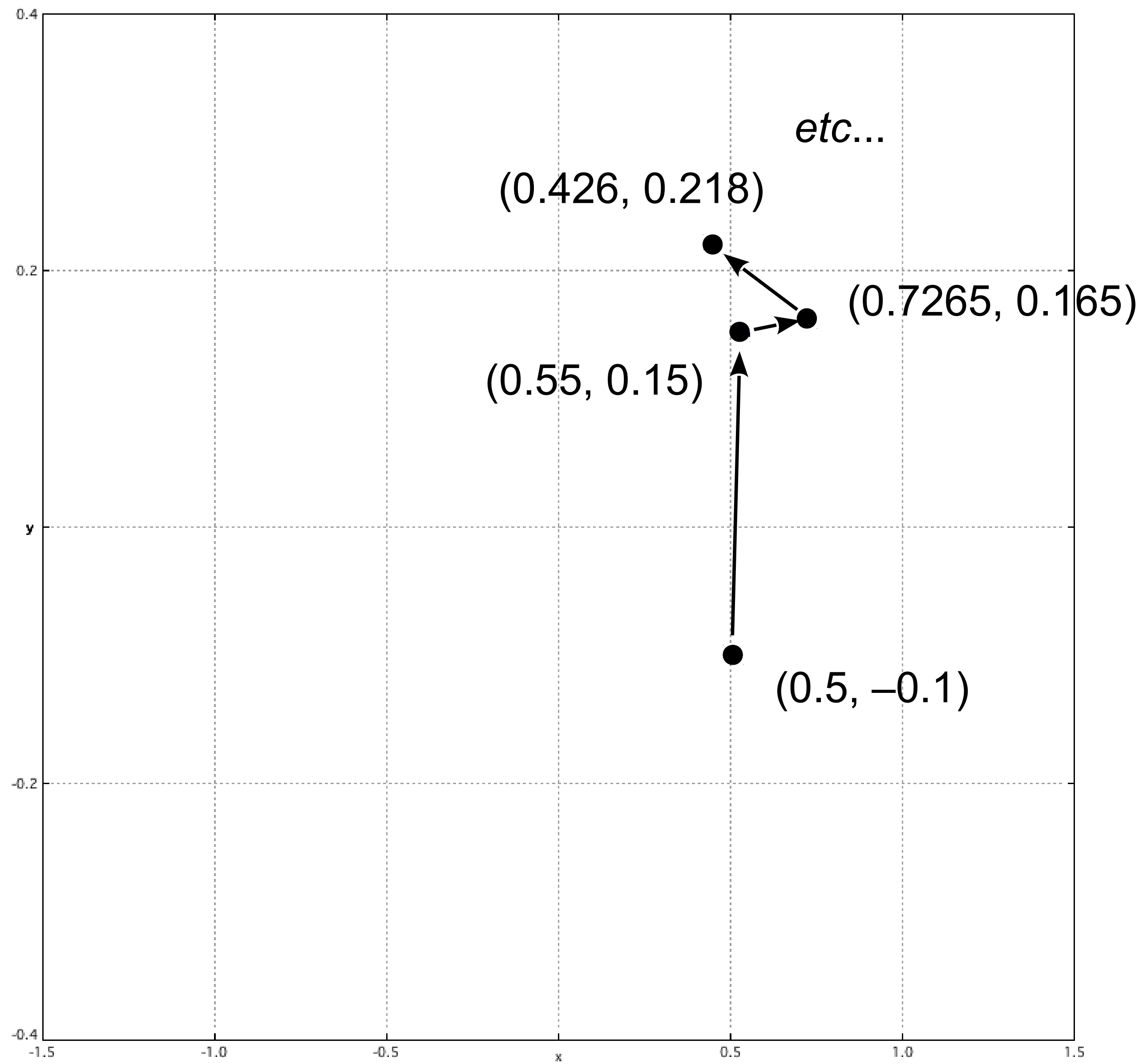
- $A = 1.4,\ B = 0.3$ (analogous to the logistic map R parameter)
- The form of the equations given in *CBN* Chapter 11 are slightly different, but mathematically equivalent to the above equations
- We start with initial values of x and y and iterate the equations to generate a trajectory, just like with the logistic map
- Long-term behavior is a **strange attractor**

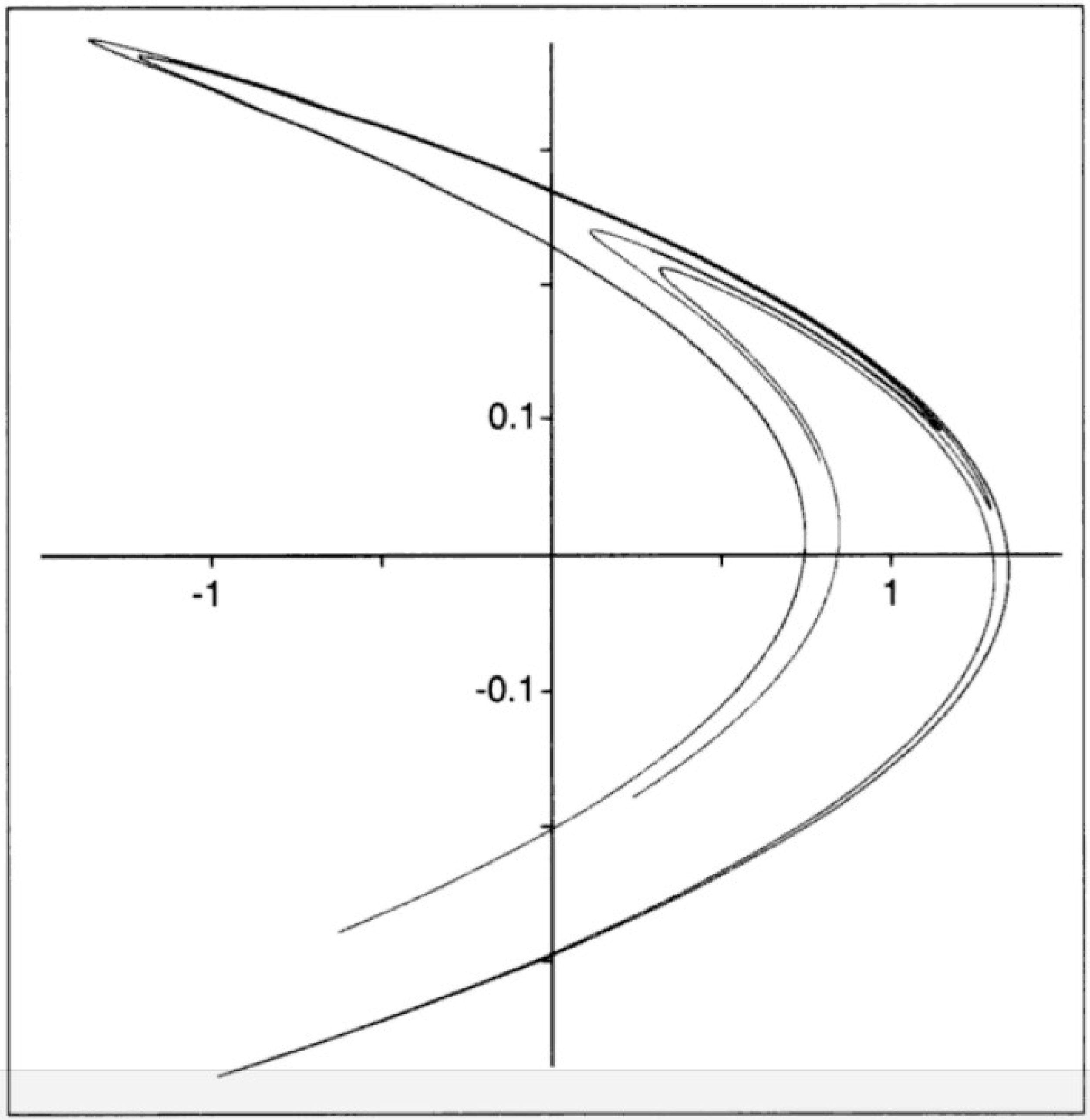
The Henon Map



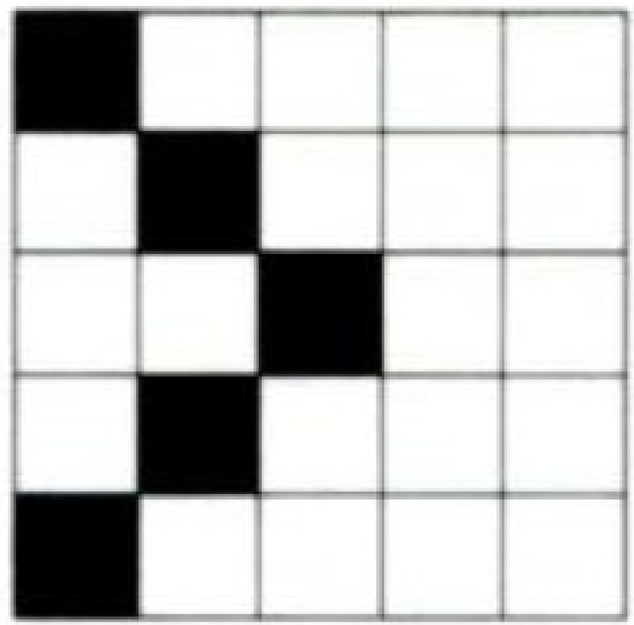
The Henon Map



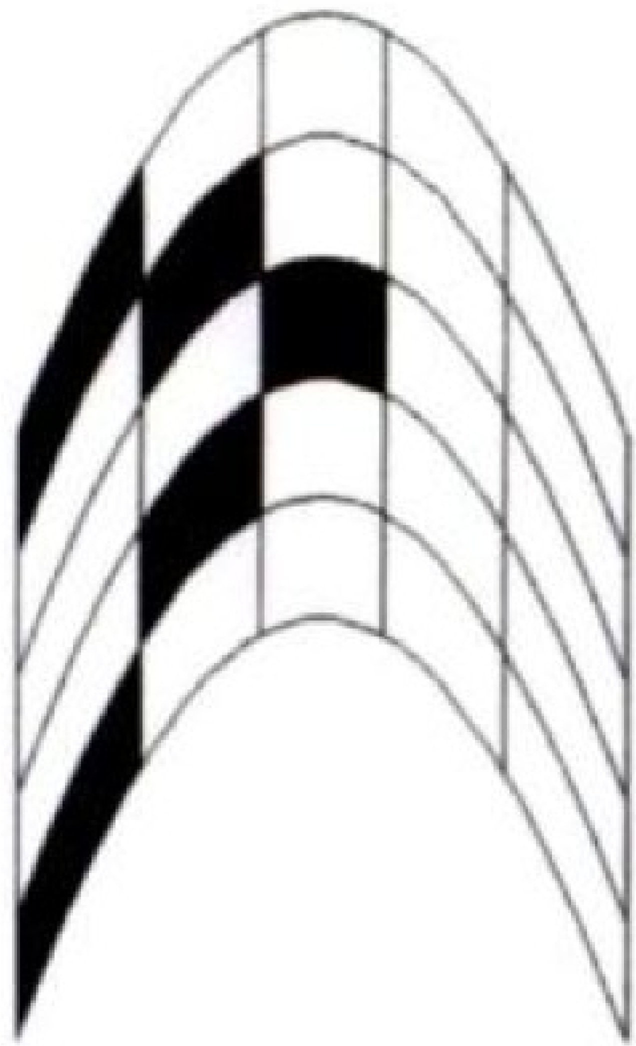
The Henon Attractor



How the Space Gets Transformed

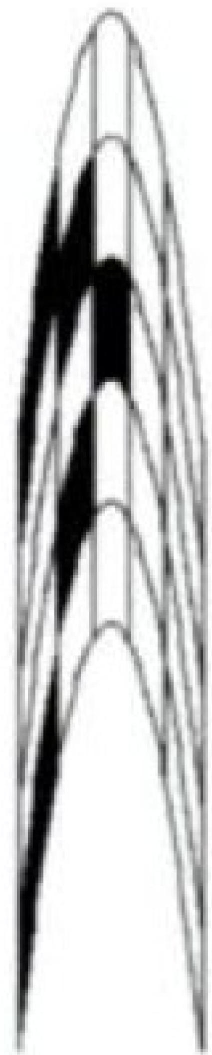


How the Space Gets Transformed



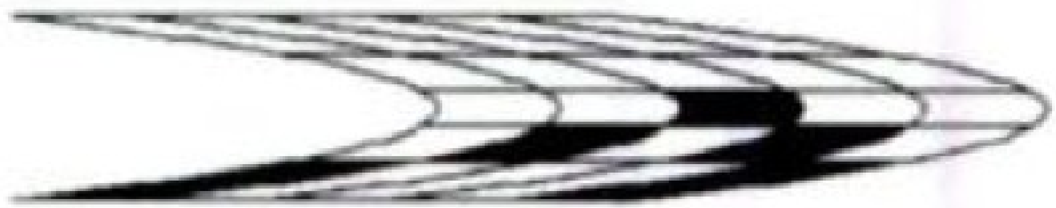
Bend up: $(x, y) \rightarrow (x, 1 - Ax^2 + y)$

How the Space Gets Transformed



Contract: $(x, y) \rightarrow (Bx, y)$

How the Space Gets Transformed



Reflect: $(x, y) \rightarrow (y, x)$

Stretching and Folding a Square Region

$t = 0$

Stretching and Folding a Square Region

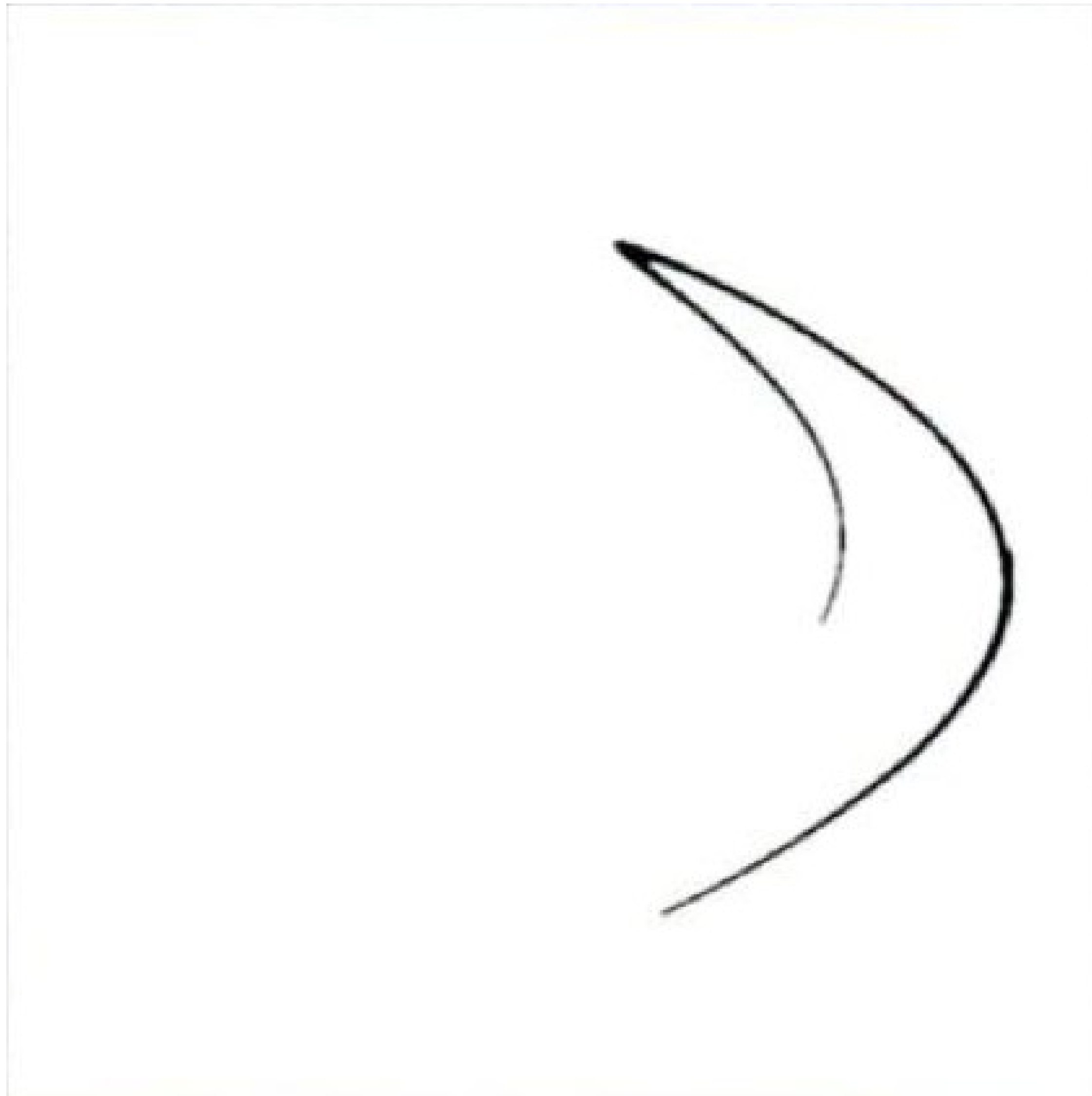
$t = 1$

Stretching and Folding a Square Region

$t = 2$

Stretching and Folding a Square Region

$t = 3$



Stretching and Folding a Square Region

$t = 5$

Stretching and Folding a Square Region

$t = 10$

Stretching and Folding the Attractor Itself

Starting shape:

Stretching and Folding the Attractor Itself

Starting shape:

Bend up

Stretching and Folding the Attractor Itself

Starting shape:

Bend up

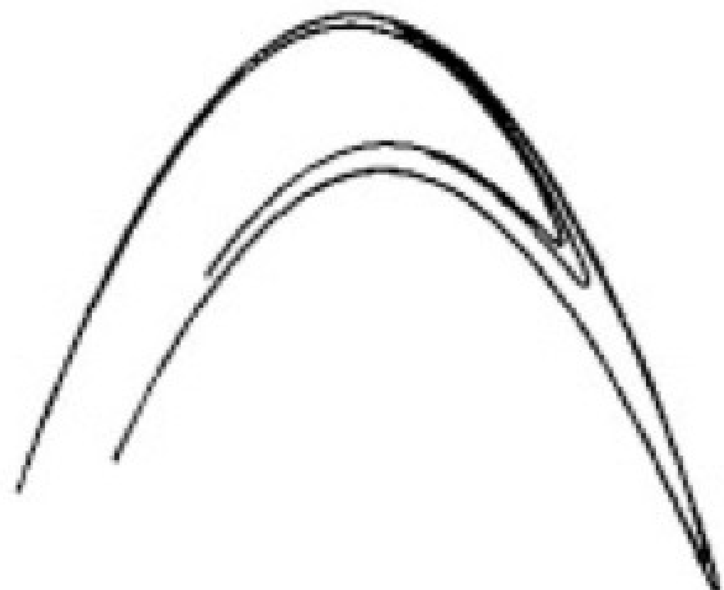
Stretching and Folding the Attractor Itself

Starting shape:

Bend up

Stretching and Folding the Attractor Itself

Starting shape:



Bend up

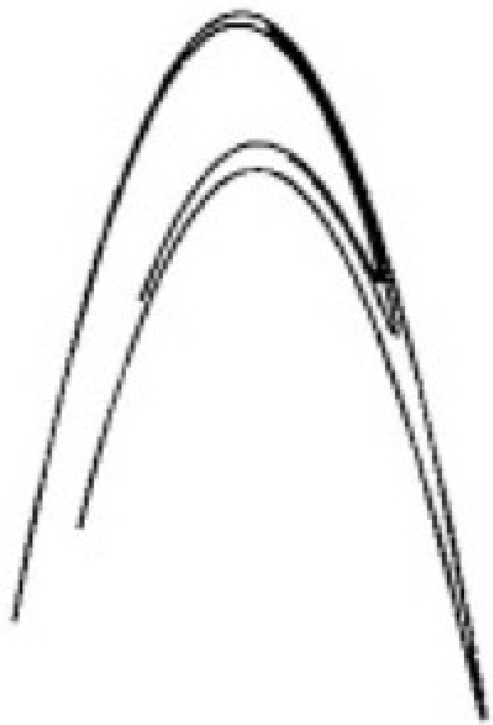
Stretching and Folding the Attractor Itself

Starting shape:

Bend up

Stretching and Folding the Attractor Itself

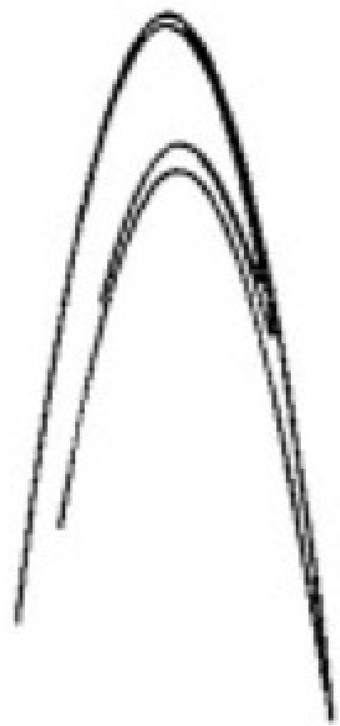
Starting shape:



Contract

Stretching and Folding the Attractor Itself

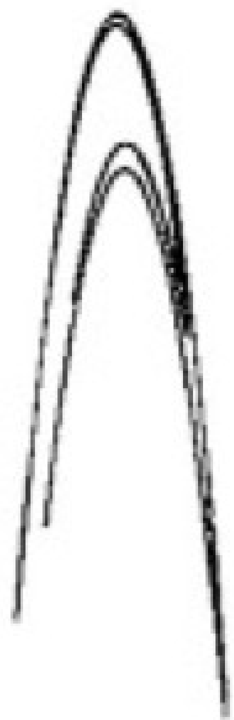
Starting shape:



Contract

Stretching and Folding the Attractor Itself

Starting shape:



Contract

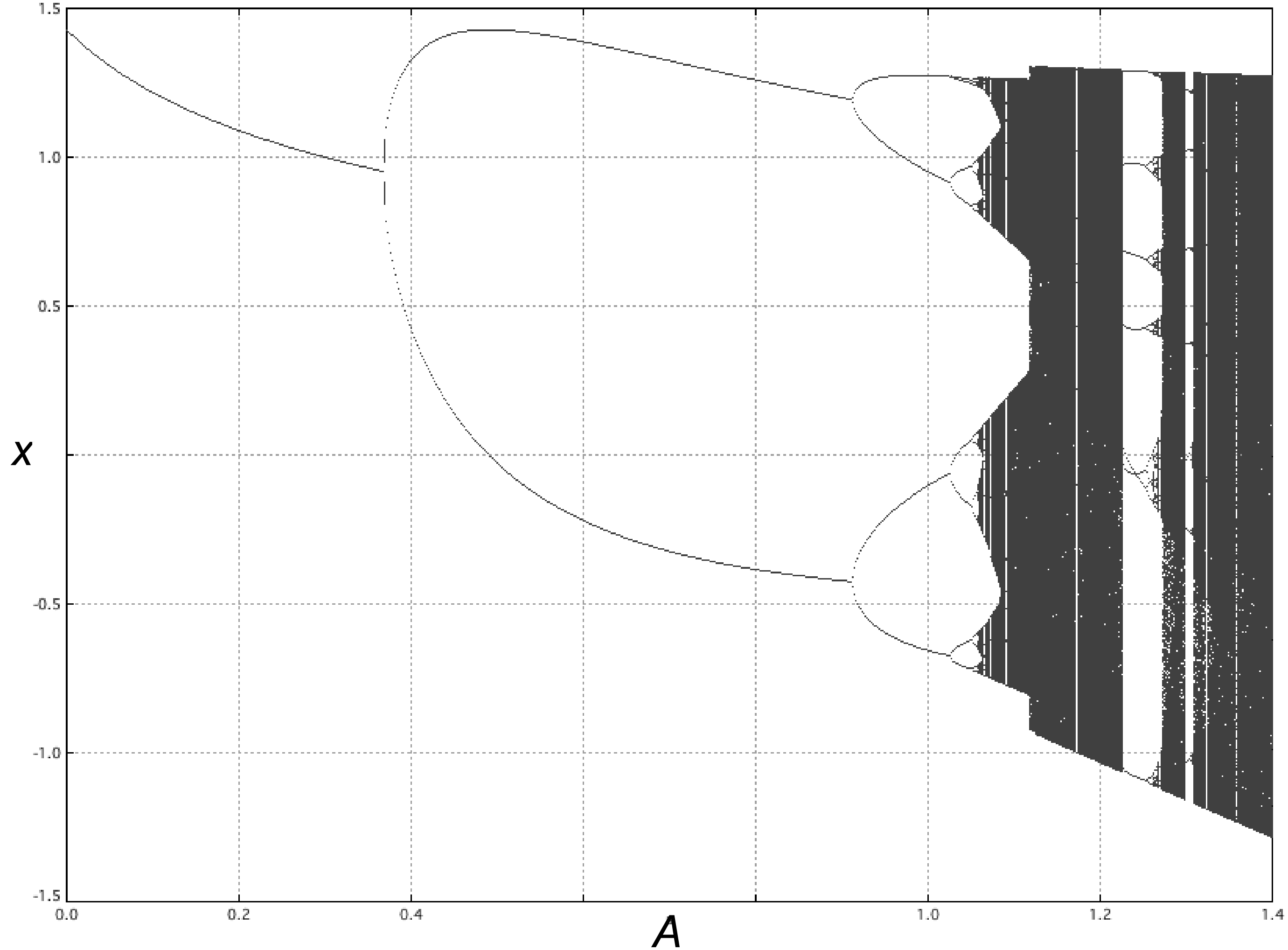
Stretching and Folding the Attractor Itself

Starting shape:

Reflect

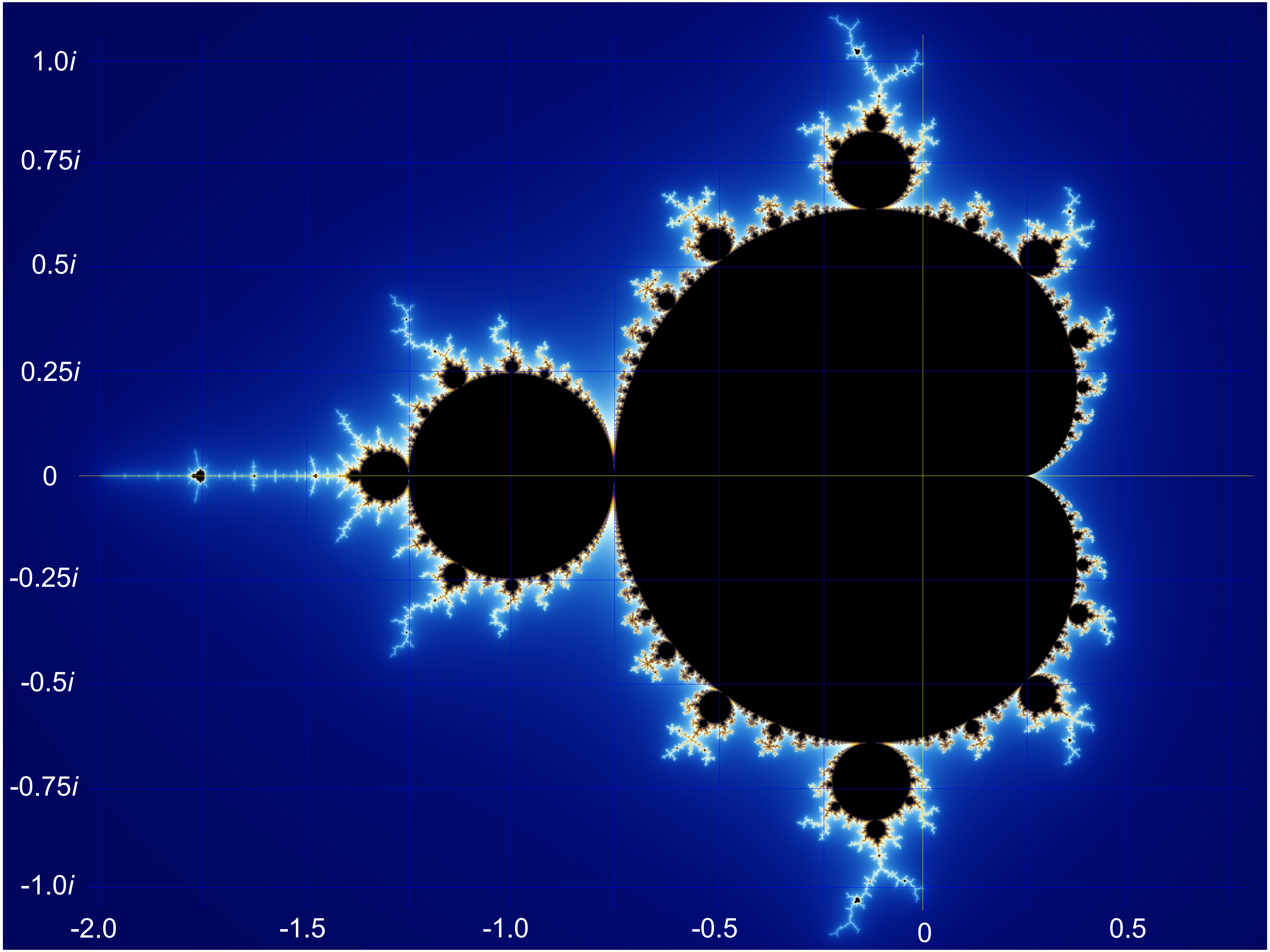
The Henon Attractor

- Fractal structure of attractor
- Bifurcation diagram for parameter *A* (holding *B* = 0.3 constant)

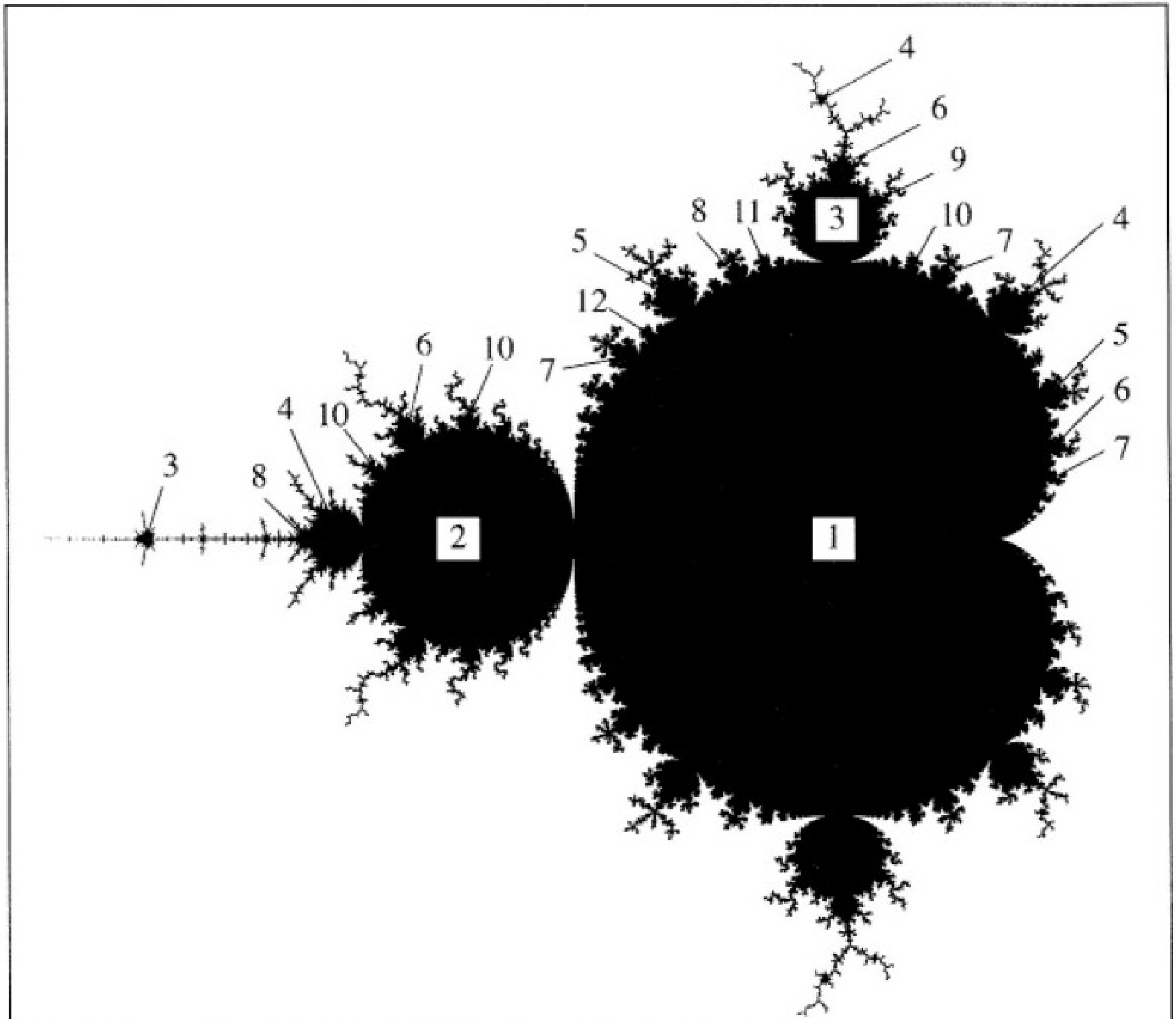


The Mandelbrot Set

The Mandelbrot Set



Regions of Periodic Behavior



A Surprising Correspondence